Path-transformations in probability and representation theory

Neil O'Connell

University of Warwick

Infinite-dimensional analysis and representation theory
Bielefeld, December 2007

Joint work with Philippe Biane and Philippe Bougerol

Pitman's Theorem (1975)

If $(B(t), t \geq 0)$ is a one-dimensional Brownian motion, then

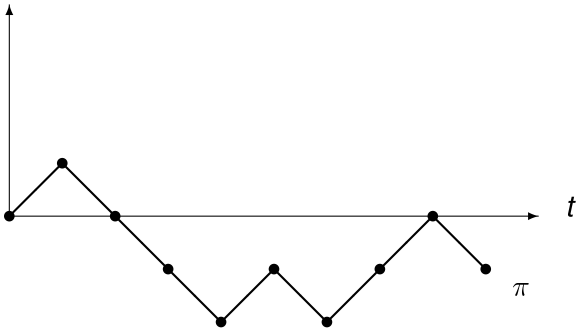
$$B(t) - 2 \inf_{s \leq t} B(s), \qquad t \geq 0$$

is a three-dimensional Bessel process.

The Pitman transform

For continuous $\pi : [0, T] \to \mathbb{R}$ with $\pi(0) = 0$, define $\mathcal{P}\pi$ by

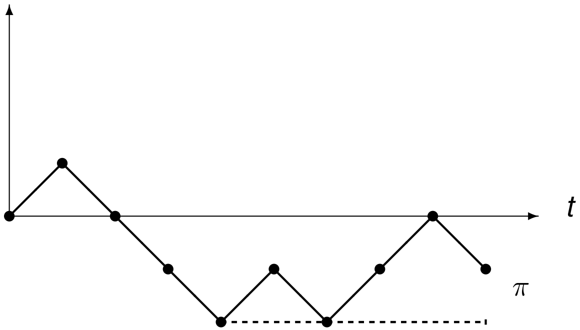
$$\mathcal{P}\pi(t) = \pi(t) - 2 \inf_{s \le t} \pi(s).$$



The Pitman transform

For continuous $\pi : [0, T] \to \mathbb{R}$ with $\pi(0) = 0$, define $\mathcal{P}\pi$ by

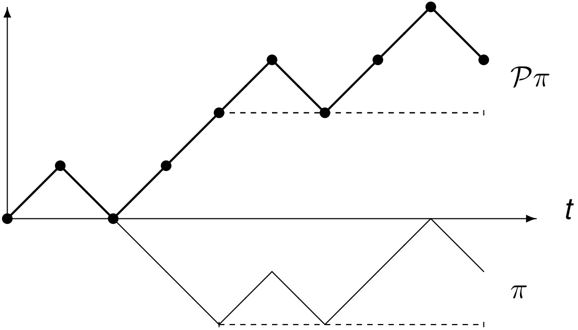
$$\mathcal{P}\pi(t) = \pi(t) - 2 \inf_{s \le t} \pi(s).$$



The Pitman transform

For continuous $\pi : [0, T] \to \mathbb{R}$ with $\pi(0) = 0$, define $\mathcal{P}\pi$ by

$$\mathcal{P}\pi(t) = \pi(t) - 2 \inf_{s \leq t} \pi(s).$$



Generalised Pitman Transforms

Let V be a finite-dimensional Euclidean space.

Generalised Pitman Transforms

Let V be a finite-dimensional Euclidean space.

$C_0^T(V) = \{$ continuous paths $\eta : [0, T] \to V$ with $\eta(0) = 0$ $\}$.

Generalised Pitman Transforms

Let V be a finite-dimensional Euclidean space.

$C_0^T(V) = \{$ continuous paths $\eta : [0, T] \to V$ with $\eta(0) = 0$ $\}$.

For $\alpha \in V$, set $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

Generalised Pitman Transforms

Let V be a finite-dimensional Euclidean space.

$C_0^T(V) = \{$ continuous paths $\eta : [0, T] \to V$ with $\eta(0) = 0$ $\}$.

For $\alpha \in V$, set $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

For $\eta \in C_0^T(V)$ and $\alpha \in V$, define

$$\mathcal{P}_\alpha \eta(t) = \eta(t) - \inf_{s \le t} \alpha^\vee(\eta(s))\alpha, \qquad 0 \le t \le T.$$

Hyperplane reflections and the braid relations

For $\alpha \in V$, let s_α denote the reflection through $\alpha^\perp$:

$$s_\alpha \lambda = \lambda - \alpha^\vee(\lambda)\alpha.$$

Hyperplane reflections and the braid relations

For $\alpha \in V$, let s_α denote the reflection through $\alpha^\perp$:

$$s_\alpha \lambda = \lambda - \alpha^\vee(\lambda)\alpha.$$

Let $\beta \in V$ with $(\alpha, \beta) = -\cos(\pi/n)$. Then

$$s_\alpha s_\beta s_\alpha \cdots = s_\beta s_\alpha s_\beta \cdots \qquad n \text{ terms}$$

Hyperplane reflections and the braid relations

For $\alpha \in V$, let s_α denote the reflection through $\alpha^\perp$:

$$s_\alpha \lambda = \lambda - \alpha^\vee(\lambda)\alpha.$$

Let $\beta \in V$ with $(\alpha, \beta) = -\cos(\pi/n)$. Then

$$s_\alpha s_\beta s_\alpha \cdots = s_\beta s_\alpha s_\beta \cdots \qquad n \text{ terms}$$

Theorem:

$$\mathcal{P}_\alpha \mathcal{P}_\beta \mathcal{P}_\alpha \cdots = \mathcal{P}_\beta \mathcal{P}_\alpha \mathcal{P}_\beta \cdots \qquad n \text{ terms}$$

Finite Coxeter groups

$W =$ finite group of isometries on V
$S = \{s_\alpha, \alpha \in \Delta\}$ generating set of 'simple' reflections

Finite Coxeter groups

$W =$ finite group of isometries on V
$S = \{s_\alpha, \alpha \in \Delta\}$ generating set of 'simple' reflections

If $\pi/n_{\alpha\beta}$ is the angle between hyperplanes $\alpha^\perp$ and $\beta^\perp$ then

$$s_\alpha^2 = 1 \qquad (s_\alpha s_\beta)^{n_{\alpha\beta}} = 1 \qquad \alpha, \beta \in \Delta$$

are the defining relations for W.

Finite Coxeter groups

W = finite group of isometries on V
$S = \{s_\alpha, \alpha \in \Delta\}$ generating set of 'simple' reflections

If $\pi/n_{\alpha\beta}$ is the angle between hyperplanes $\alpha^\perp$ and $\beta^\perp$ then

$$s_\alpha^2 = 1 \qquad (s_\alpha s_\beta)^{n_{\alpha\beta}} = 1 \qquad \alpha, \beta \in \Delta$$

are the defining relations for W.

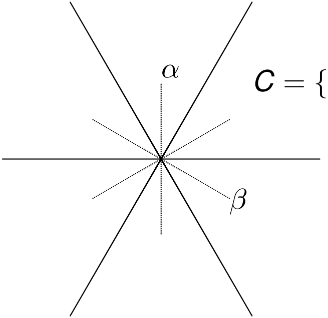
The (closure of)

$$C = \{\lambda \in V : \ (\alpha, \lambda) > 0, \ \forall \alpha \in \Delta\}$$

is a fundamental domain for the action of W on V.

Example

$$\alpha = (e_1 - e_2)/\sqrt{2} \qquad W = \langle\, s_\alpha,\, s_\beta \,\rangle \simeq S_3 = \langle\, (12), (23) \,\rangle$$
$$\beta = (e_2 - e_3)/\sqrt{2}$$



$$C = \{x_1 > x_2 > x_3\}$$

Generalised Pitman Transforms II

For each $w \in W$, we can define

$$\mathcal{P}_w = \mathcal{P}_{\alpha_1} \cdots \mathcal{P}_{\alpha_k}$$

where $w = s_{\alpha_1} \cdots s_{\alpha_k}$ is *any* reduced decomposition of w.

The longest element

Let W be a finite Coxeter group with generating simple reflections $S = \{s_\alpha, \alpha \in \Delta\}$. The length of an element $w \in W$ is the minimal number of terms required to write w as a product of simple reflections. There is a unique $w_0 \in W$ of maximal length.

For example, the longest element in S_3 is

$$(13) = (12)(23)(12) = (23)(12)(23).$$

Some properties of $\mathcal{P}_{w_0}$

- $\mathcal{P}_{w_0}^2 = \mathcal{P}_{w_0}$

Some properties of $\mathcal{P}_{w_0}$

- $\mathcal{P}_{w_0}^2 = \mathcal{P}_{w_0}$

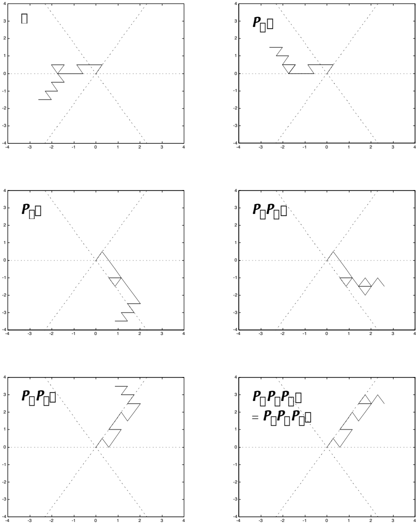
Say $\pi \in C_0^T(V)$ is *dominant* if $\pi(s) \in \overline{C}$ for all $s \le T$.

Some properties of $\mathcal{P}_{w_0}$

- $\mathcal{P}_{w_0}^2 = \mathcal{P}_{w_0}$

Say $\pi \in C_0^T(V)$ is *dominant* if $\pi(s) \in \overline{C}$ for all $s \leq T$.

- For any $\eta \in C_0^T(V)$, $\mathcal{P}_{w_0}\eta$ is dominant.



Brownian motion (conditioned to stay) in a cone

Let C be a convex cone in V.

Brownian motion (conditioned to stay) in a cone

Let C be a convex cone in V.

Let $p_t^0(x, y)$ be the heat kernel on C with Dirichlet boundary conditions.

Brownian motion (conditioned to stay) in a cone

Let C be a convex cone in V.

Let $p_t^0(x, y)$ be the heat kernel on C with Dirichlet boundary conditions.

Biane (1993): There exists a unique (up to constant factors) positive p^0-harmonic function h on C.

Brownian motion (conditioned to stay) in a cone

Let C be a convex cone in V.

Let $p_t^0(x,y)$ be the heat kernel on C with Dirichlet boundary conditions.

Biane (1993): There exists a unique (up to constant factors) positive p^0-harmonic function h on C.

Brownian motion in the cone C is defined to be the corresponding Doob h-transform, with infinitessimal generator and transition density given respectively by

$$\frac{1}{2}\Delta + \nabla(\log h)\cdot\nabla \qquad q_t(x,y) = \frac{h(y)}{h(x)} p_t^0(x,y).$$

The three-dimensional Bessel Process

If $V = \mathbb{R}$ and $C = \mathbb{R}_+$ then

$$p_t^0(x,y) = p_t(x,y) - p_t(x,-y) \qquad h(x) = x.$$

Brownian motion in $\mathbb{R}_+$ is the three-dimensional Bessel process, with infinitessimal generator

$$\frac{1}{2}\frac{d}{dx^2} + \frac{1}{x}\frac{d}{dx}.$$

Brownian motion in a Weyl chamber

Let W be a finite Coxeter group acting on V with fundamental chamber C. Then

$$p_t^0(x,y) = \sum_{w \in W} \varepsilon(w) p_t(x, wy) \qquad h(x) = \prod_{\alpha \in \Phi^+} (\alpha, x).$$

Brownian motion in a Weyl chamber

Let W be a finite Coxeter group acting on V with fundamental chamber C. Then

$$p_t^0(x,y) = \sum_{w\in W} \varepsilon(w) p_t(x, wy) \qquad h(x) = \prod_{\alpha\in\Phi^+} (\alpha, x).$$

If $W = S_n$ and $V = \mathbb{R}^n$, the Brownian motion in C is distributed as the eigenvalue process of a Brownian motion in the Lie algebra of $n \times n$ Hermitian matrices.

A generalisation of Pitman's theorem

Let W be a finite Coxeter group acting on V with fundamental chamber C.

Theorem. If η is a Brownian motion in V (with respect to some probability $\mathbb{P}$ on $C_0^T(V)$), then $\mathcal{P}_{w_0}\eta$ is a Brownian motion in C.

Representation-theoretic proof in Weyl group case; proof for the general case uses duality properties of $\mathcal{P}_{w_0}$.

String parameters

cf. Littelmann (1998), Berenstein and Zelevinsky (2001)

Reduced decomposition $w_0 = s_{\alpha_1} \dots s_{\alpha_q}$

Set $\eta_q = \eta$ and, for $i \leq q$,

$$\eta_{i-1} = \mathcal{P}_{\alpha_i} \dots \mathcal{P}_{\alpha_q} \eta \qquad x_i = -\inf_{T \geq t \geq 0} \alpha_i^\vee(\eta_i(t)).$$

String parameters

cf. Littelmann (1998), Berenstein and Zelevinsky (2001)

Reduced decomposition $w_0 = s_{\alpha_1} \dots s_{\alpha_q}$

Set $\eta_q = \eta$ and, for $i \leq q$,

$$\eta_{i-1} = \mathcal{P}_{\alpha_i} \dots \mathcal{P}_{\alpha_q} \eta \qquad x_i = -\inf_{T \geq t \geq 0} \alpha_i^\vee(\eta_i(t)).$$

Then $\eta_0 = \mathcal{P}_{w_0} \eta$ and

$$\mathcal{P}_{w_0} \eta(T) = \eta(T) + \sum_{i=1}^{q} x_i \alpha_i.$$

String parameters

cf. Littelmann (1998), Berenstein and Zelevinsky (2001)

Reduced decomposition $w_0 = s_{\alpha_1} \dots s_{\alpha_q}$

Set $\eta_q = \eta$ and, for $i \leq q$,

$$\eta_{i-1} = \mathcal{P}_{\alpha_i} \dots \mathcal{P}_{\alpha_q} \eta \qquad x_i = - \inf_{T \geq t \geq 0} \alpha_i^\vee(\eta_i(t)).$$

Then $\eta_0 = \mathcal{P}_{w_0} \eta$ and

$$\mathcal{P}_{w_0} \eta(T) = \eta(T) + \sum_{i=1}^{q} x_i \alpha_i.$$

Let $\mathbf{i} = (\alpha_1, \dots, \alpha_q)$ and $\varrho_{\mathbf{i}}(\eta) = (x_1, \dots, x_q)$.

String parameters

- The mapping

$$\mathcal{P}_{w_0} \times \varrho_{\mathbf{i}} : C_0^T(V) \to \bigcup_{\lambda \in \overline{C}} D^\lambda \times M_{\mathbf{i}}^\lambda$$

is a bijection, where D^λ is the set of dominant paths π with $\pi(T) = \lambda$ and $M_{\mathbf{i}}^\lambda$ is a (generalised) 'string polytope'

String parameters

- The mapping

$$\mathcal{P}_{w_0} \times \varrho_{\mathbf{i}} : C_0^T(V) \to \bigcup_{\lambda \in \overline{C}} D^\lambda \times M_{\mathbf{i}}^\lambda$$

  is a bijection, where D^λ is the set of dominant paths π with $\pi(T) = \lambda$ and $M_{\mathbf{i}}^\lambda$ is a (generalised) 'string polytope'
- In the type A case, for suitable $\mathbf{i}$, $M_{\mathbf{i}}^\lambda$ is essentially the set of Gelfand-Tsetlin patterns with bottom row λ and the restricton of $\mathcal{P}_{w_0} \times \varrho_{\mathbf{i}}$ to 'lattice paths' is equivalent to RSK:

$$D^\lambda \simeq \text{standard tableaux with shape } \lambda$$

$$M_{\mathbf{i}}^\lambda \simeq \text{semistandard tableaux with shape } \lambda$$

Littelmann's path model

Suppose W is a Weyl group.

Let $P = \{\mu \in V : \alpha^\vee(\mu) \in \mathbb{Z}, \ \forall \alpha \in \Delta\}$, $P_+ = P \cap \overline{C}$

Let π be dominant with $\pi(T) = \lambda \in P_+$ and *integral:*

$$\inf_{s \leq T} \alpha^\vee(\pi(s)) \in \mathbb{Z} \qquad \forall \alpha \text{ simple.}$$

Raising/lowering operators e_α, f_α generate path module B_π

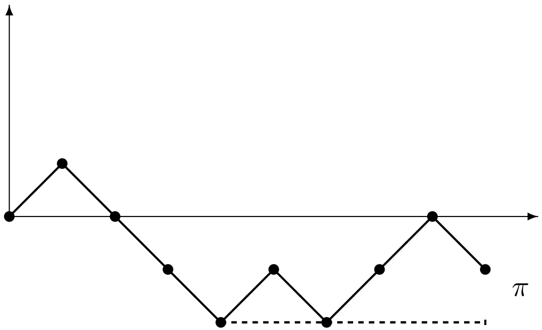
Then, for example,

$$\dim \lambda = |B_\pi|, \qquad M^\lambda_\mu = |\{\eta \in B_\pi : \ \eta(T) = \mu\}|.$$

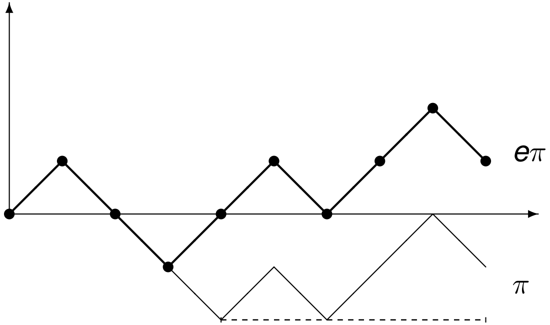
Similar formulae for the Littlewood-Richardson coefficients.

From the definition, $\mathcal{P}_\alpha = e_\alpha^{\mathbf{MAX}}$.

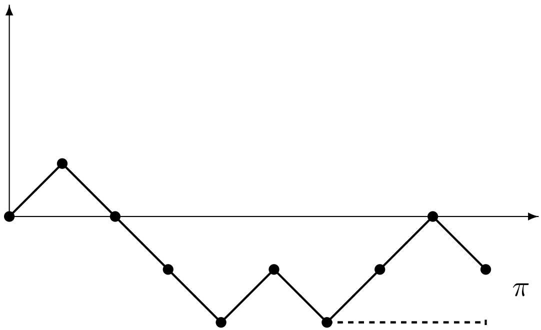
Littelmann's raising and lowering operators



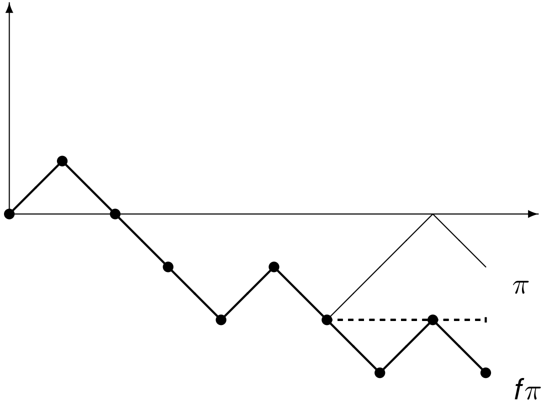
Littelmann's raising and lowering operators



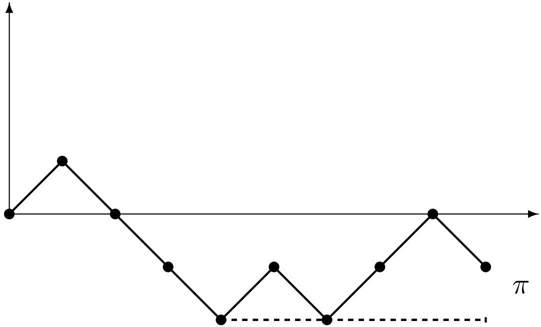
Littelmann's raising and lowering operators



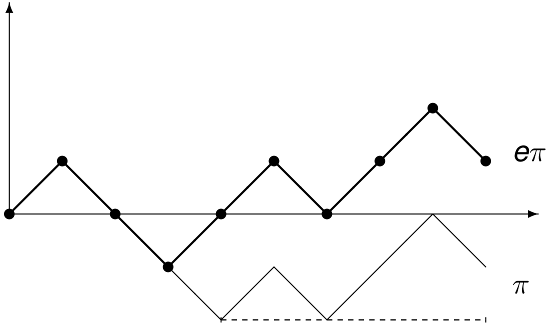
Littelmann's raising and lowering operators



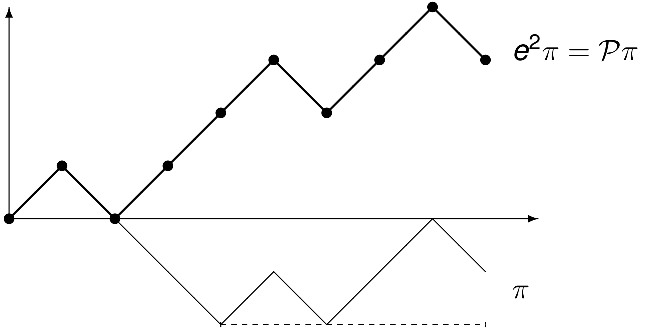
Connection with the Pitman transform



Connection with the Pitman transform



Connection with the Pitman transform



String polytopes

Theorem. Setting $L_\pi := \mathcal{P}_{w_0}^{-1}\pi$ we have

$$B_\pi = \{\eta \in L_\pi :\ \varrho_{\mathbf{i}}(\eta) \in \mathbb{N}^q\}.$$

String polytopes

Theorem. Setting $L_\pi := \mathcal{P}_{w_0}^{-1}\pi$ we have

$$B_\pi = \{\eta \in L_\pi : \ \varrho_{\mathbf{i}}(\eta) \in \mathbb{N}^q\}.$$

Littelmann (1998) showed that

$$\varrho_{\mathbf{i}}(B_\pi) = C_{\mathbf{i}} \cap \mathbb{N}^q \cap K_\pi$$

where

$$K_\pi = \left\{ x \in \mathbb{R}_+^q : \ 0 \le x_i \le \alpha_i^\vee \left(\pi(T) - \sum_{j=1}^{i-1} x_j \alpha_j \right), \ i = 1, \ldots, q \right\},$$

and $C_{\mathbf{i}}$ is a convex polyhedral cone in $\mathbb{R}^q$. Berenstein and Zelevinsky (2001) give an explicit description of $C_{\mathbf{i}}$.

String polytopes

- We show that in the general case, if $\pi(T) = \lambda$,

$$M_{\mathbf{i}}^{\lambda} := \varrho_{\mathbf{i}}(L_\pi) = C_{\mathbf{i}} \cap K_\pi$$

  with K_π as before and $C_{\mathbf{i}}$ a convex polyhedral cone in $\mathbb{R}^q$. In particular, $\varrho_{\mathbf{i}}(L_\pi)$ depends only on the endpoint λ.

String polytopes

- We show that in the general case, if $\pi(T) = \lambda$,

$$M_{\mathbf{i}}^{\lambda} := \varrho_{\mathbf{i}}(L_\pi) = C_{\mathbf{i}} \cap K_\pi$$

  with K_π as before and $C_{\mathbf{i}}$ a convex polyhedral cone in $\mathbb{R}^q$. In particular, $\varrho_{\mathbf{i}}(L_\pi)$ depends only on the endpoint λ.
- Moreover, as in the Weyl goup case, the string parameters corresponding to one decomposition determine those of another via a piecewise linear continuous map (which does not depend on the path).

Duistermaat-Heckman measure

Theorem. Let η be a standard Brownian motion in V.

Duistermaat-Heckman measure

Theorem. Let η be a standard Brownian motion in V.

- The conditional law of $\varrho_{\mathbf{i}}(\eta)$, given $(\mathcal{P}_{w_0}\eta(s), s \leq T)$ and $\mathcal{P}_{w_0}\eta(T) = \lambda$, is almost surely uniform on $M_{\mathbf{i}}^{\lambda}$.

Duistermaat-Heckman measure

Theorem. Let η be a standard Brownian motion in V.

- The conditional law of $\varrho_{\mathbf{i}}(\eta)$, given $(\mathcal{P}_{w_0}\eta(s), s \le T)$ and $\mathcal{P}_{w_0}\eta(T) = \lambda$, is almost surely uniform on $M_{\mathbf{i}}^{\lambda}$.
- The conditional law of $\eta(T)$, given $(\mathcal{P}_{w_0}\eta(s), s \le T)$ and $\mathcal{P}_{w_0}\eta(T) = \lambda$, depends only on λ; denoting this law by μ_{DH}^{λ},

$$\int_V e^{(z,v)} \mu_{DH}^{\lambda}(dv) = k \frac{\sum_{w \in W} (-1)^w e^{(wz,\lambda)}}{h(z)h(\lambda)} \qquad z \in V^*.$$

Duistermaat-Heckman measure

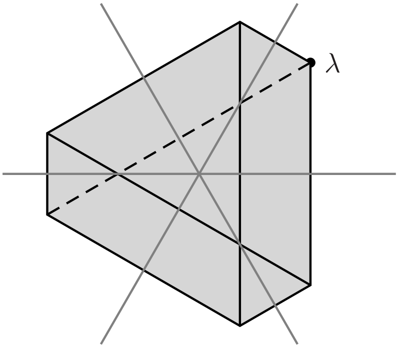
Theorem. Let η be a standard Brownian motion in V.

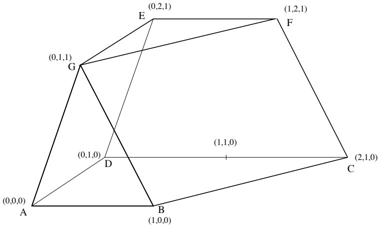
- The conditional law of $\varrho_{\mathbf{i}}(\eta)$, given $(\mathcal{P}_{w_0}\eta(s), s \le T)$ and $\mathcal{P}_{w_0}\eta(T) = \lambda$, is almost surely uniform on $M_{\mathbf{i}}^{\lambda}$.
- The conditional law of $\eta(T)$, given $(\mathcal{P}_{w_0}\eta(s), s \le T)$ and $\mathcal{P}_{w_0}\eta(T) = \lambda$, depends only on λ; denoting this law by μ_{DH}^{λ},

$$\int_V e^{(z,v)} \mu_{DH}^{\lambda}(dv) = k \frac{\sum_{w \in W} (-1)^w e^{(wz,\lambda)}}{h(z)h(\lambda)} \qquad z \in V^*.$$

- μ_{DH}^{λ} is supported on the convex hull of $W\lambda$ and has a continuous, piecewise polynomial density.

DH measure for A_2



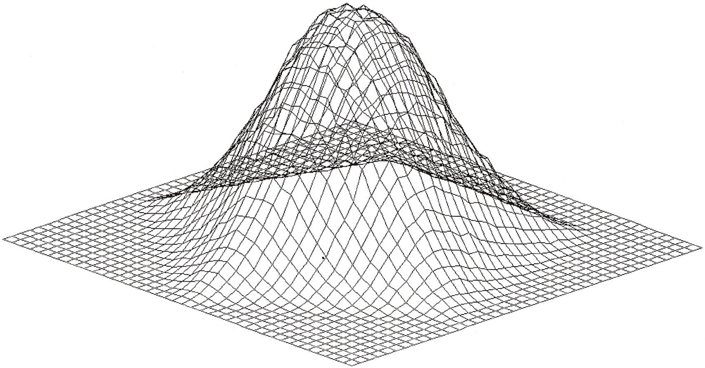


From:

Valery Alexeev, Michel Brion. Toric degenerations of spherical varieties. (math.AG/0403379), and

S. Morier-Genoud. Geometric lifting of the canonical basis and semitoric degenerations of Richardson varieties. (math.RT/0504538).

DH measure for $I(5)$ (with $\lambda \in \partial C$)



Further results

- Generalisation of Greene's formula

Further results

- Generalisation of Greene's formula
- Duality properties, connections with queueing theory

Further results

- Generalisation of Greene's formula
- Duality properties, connections with queueing theory
- Crystal / plactic monoid structure

Further results

- Generalisation of Greene's formula
- Duality properties, connections with queueing theory
- Crystal / plactic monoid structure
- Geometric lifting / tropicalisation

References

A. Berenstein and A. Zelevinsky. Tensor product multiplicities, canonical bases and totally positive varieties. *Invent. Math.* 143 (2001), no. 1, 77–128.

Ph. Biane, Ph. Bougerol, N. O'Connell. Littelmann paths and Brownian paths. *Duke Math. J.* 130 (2005) 127–167.

Ph. Biane, Ph. Bougerol, N. O'Connell. Continuous crystals and Duistermaat-Heckman measure for Coxeter groups. In preparation.

G. J. Heckman. Projections of orbits and asymptotic behavior of multiplicities for compact connected Lie groups. *Invent. Math.* 67 (1982), no. 2, 333–356.

P. Littelmann. Cones, crystals, and patterns. *Transform. Groups* 3 (1998), no. 2, 145–179.

J.W. Pitman. One-dimensional Brownian motion and the three-dimensional Bessel process. *Adv. Appl. Probab.* 7 (1975) 511-526.