Some new perspectives on moments of random matrices

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Based on joint work with F. D. Cunden, F. Mezzadri and N. Simm

Random matrices: a brief history

1930's — Multivariate statistics (Wishart, . . .)

1950's — Nuclear physics (Wigner, . . .)

1960's — Local statistics of eigenvalues, symmetry (Dyson, . . .)

1980's — Enumerative geometry (Harer-Zagier, . . .)

2000's — Combinatorial representation theory and related models in statistical physics (Baik-Deift-Johansson, . . .)

Gaussian random matrices

Probability measure on the set of $n \times n$ Hermitian matrices:

$$P_n(X)dX = \frac{1}{2^{n/2}} \frac{1}{\pi^{n^2/2}} e^{-\frac{1}{2}\operatorname{tr} X^2} dX$$
$$= \prod_{1 \le i \le n} \frac{1}{\sqrt{2\pi}} e^{-x_{ii}^2/2} dx_{ii} \prod_{1 \le i < j \le n} \frac{1}{\pi} e^{-|x_{ij}|^2} d\Re x_{ij} d\Im x_{ij}$$

Joint law (density) of the eigenvalues:

$$p_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{1!2! \cdots n!} \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^{n} \frac{e^{-\lambda_i^2/2}}{\sqrt{2\pi}} \quad (\lambda_i \in \mathbb{R})$$

Moments

$$\mathbb{E}\,\mathrm{tr}\,X^{2k} = \int_{\mathbb{R}^n} \left(\sum_{i=1}^{n} \lambda_i^{2k} \right) p_n(\lambda_1, \ldots, \lambda_n) d\lambda_1 \cdots d\lambda_n$$

$$\begin{aligned}
\mathbb{E}\,\mathrm{tr}\,X^2 &= n^2 \\
\mathbb{E}\,\mathrm{tr}\,X^4 &= 2n^3 + n \\
\mathbb{E}\,\mathrm{tr}\,X^6 &= 5n^4 + 10n^2 \\
\mathbb{E}\,\mathrm{tr}\,X^8 &= 14n^5 + 70n^3 + 21n
\end{aligned}$$

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$2 = \#$ pairwise gluings of edges of a square to obtain a sphere
$1 = \#$ pairwise gluings of edges of a square to obtain a torus

Moments

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$14 = \#$ pairwise gluings of edges of octagon to obtain a sphere
$70 = \#$ pairwise gluings of edges of octagon to obtain a torus
$21 = \#$ pairwise gluings … to obtain an orientable genus-2 surface

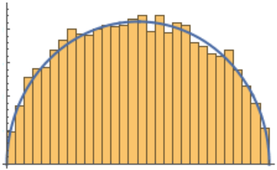
Reference: Harer-Zagier (1986)

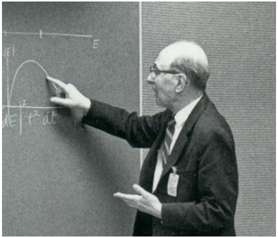
Wigner's semi-circle law

The leading coefficients 1, 2, 5, 14, . . . are the *Catalan numbers*:

$$\epsilon_0(k) = \frac{1}{k+1}\binom{2k}{k} = \frac{2}{\pi}\int_{-2}^{2} x^{2k}\sqrt{4-x^2}dx$$

These are the even moments of Wigner's (1950) semicircle law:





Mixed moments, topological recursion

More generally, mixed moments of the form

$$\mathbb{E}\left(\operatorname{tr} X^{k_1} \ldots \operatorname{tr} X^{k_r}\right)$$

can be computed using loop equations / topological recursion.

Recurrence relations

Let $Q_k(n) = \mathbb{E} \operatorname{tr} X^{2k}$. Then (Harer and Zagier, 1986)

$$(k+2)Q_{k+1}(n) = 2n(2k+1)Q_k(n) + k(2k+1)(2k-1)Q_{k-1}(n),$$

and (we find)

$$nQ_k(n+1) = 2(k+1)Q_k(n) + nQ_k(n-1).$$

Complex Wishart random matrices

Probability measure on the set of $n \times n$ positive definite Hermitian matrices:

$$P_n(X)dX = c_n \det X^n e^{-\operatorname{tr} X} dX.$$

Joint law (density) of the eigenvalues:

$$p_n(\lambda_1, \ldots, \lambda_n) = c'_n \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^{n} \lambda_i^n e^{-\lambda_i} \quad (\lambda_i \in \mathbb{R}_+)$$

Moments

$$\mathbb{E}\,\mathrm{tr}\,X^k = \int_{\mathbb{R}_+^n} \left(\sum_{i=1}^{n} \lambda_i^k \right) p_n(\lambda_1, \ldots, \lambda_n) d\lambda_1 \cdots d\lambda_n$$

$$\begin{aligned}
\mathbb{E}\,\mathrm{tr}\,X^0 &= n \\
\mathbb{E}\,\mathrm{tr}\,X^1 &= 2n^2 \\
\mathbb{E}\,\mathrm{tr}\,A^2 &= 6n^3 \\
\mathbb{E}\,\mathrm{tr}\,A^3 &= 22n^4 + 2n^2
\end{aligned}$$

These are genus expansions, with combinatorial interpretations as before.

Recurrence relations

Let $Q_k(n) = \mathbb{E}\,\mathrm{tr}\,X^k$. Then (Hagerup and Thorbørnsen, 2003)

$$(k+2)Q_{k+1}(n) = 3n(2k+1)Q_k(n) + (k-1)(k^2 - n^2)Q_{k-1}(n),$$

and (we find) similar recursions in n.

Marchenko-Pastur distribution

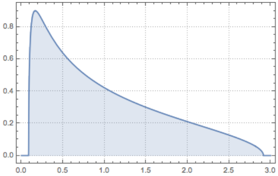
The leading coefficients 1, 2, 6, 22, … are the *Schröder numbers*

$$S_k = \int x^k p(x) dx,$$

where

$$p(x) = \frac{1}{2\pi x}\sqrt{(x-a_-)(a_+-x)}, \qquad a_\pm = (1 \pm \sqrt{2})^2$$

is the Marchenko-Pastur distribution.



Negative moments

Cunden, Mezzadri, Simm, Vivo (2015) showed that negative moments are rational functions of n:

$$\mathbb{E}\,\mathrm{tr}\,X^{-k} = \sum_{g\geq 0} \tau_{k,g}\, n^{1-k-2g}$$

and conjectured that the coefficients $\tau_{k,g}$ are positive integers.

For example,

$$\mathbb{E}\,\mathrm{tr}\,X^{-3} = \frac{6n^2}{(n^2-4)(n^2-1)} = \frac{6}{n^2} + \frac{30}{n^4} + \frac{126}{n^6} + \cdots$$

A reflection symmetry

Theorem (Cunden, Mezzadri, O'C, Simm, 2018)

$$\mathbb{E}\,\mathrm{tr}\,X^{-k-1} = \prod_{j=-k}^{k} \frac{1}{n+j}\,\mathbb{E}\,\mathrm{tr}\,X^{k}.$$

This implies the above conjecture.

Proof is straightforward using (an extension of) the Haagerup-Thorbørnsen recursion (but discovery was not straightforward!)

Average spectral zeta function

Observe that

$$\mathbb{E}\,\mathrm{tr}\,X^{-s} = \mathbb{E}\sum_{i=1}^{n} \lambda_i^{-s},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of X.

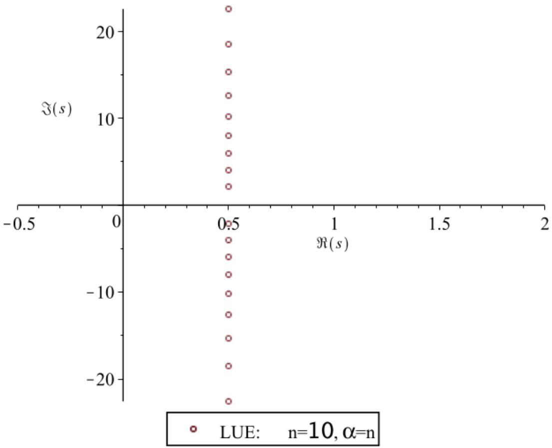
Setting

$$\xi_n(s) = \frac{1}{\Gamma(1+n-s)}\mathbb{E}\,\mathrm{tr}\,X^{-s}$$

the above reflection symmetry gives the functional equation

$$\xi_n(s) = \xi_n(1-s).$$

Natural question: where are the zeros?


20
10
ℑ(s)
−10
−20
−0.5
0
0.5
1
1.5
2
ℜ(s)
LUE: n=10, α=n

Moments as polynomials in k

n	$(2n-1)!(n-1)!\mathbb{E}\,\mathrm{tr}\,X^k/(n+k)!$
1	1
2	$6+k+k^2$
3	$120+28k+29k^2+2k^3+k^4$
4	$5040+1356k+1432k^2+153k^3+79k^4+3k^5+k^6$

As orthogonal polynomials ...

Recall

$$\xi_n(s) = \frac{1}{\Gamma(1+n-s)} \mathbb{E} \operatorname{tr} X^{-s}, \qquad \xi_n(s) = \xi_n(1-s).$$

Theorem

Let $s = 1/2 + ix$. Then

$$\xi_n(s) = \frac{1}{\Gamma(n)\Gamma(2n)} S_{n-1}\left(x^2; \frac{3}{2}, \frac{1}{2}, n + \frac{1}{2}\right),$$

where S_{n-1} is a continuous dual Hahn polynomial of degree $n-1$.

Corollary

The zeros of $\xi_n(s)$ lie on the critical line $\Re s = 1/2$.

Moments of random matrices and hypergeometric orthogonal polynomials

More generally, we find that moments $\mathbb{E}\,\mathrm{tr}\,X^k$, for classical ensembles of random matrices, are given in terms of hypergeometric (or 'neo-classical') orthogonal polynomials in the variable k.

Moreover, for the Gaussian (GUE) and Wishart (LUE) ensembles, they are also hypergeometric orthogonal polynomials in the dimension n.

The Harer-Zagier and Hagerup-Thorbørnsen recursions are in fact the three term recurrence relations for the Meixner and Hahn polynomials, or equivalently the discrete Sturm-Liouville equations for their duals; recursions in n also.

Ensemble	Classical OP	Moments
GUE	Hermite	Meixner-Pollaczek
LUE	Laguerre	Continuous dual Hahn
JUE	Jacobi	Wilson

Mellin transforms of orthogonal polynomials

Bump and Ng (1986) and Bump, Choi, Kurlberg and Vaaler (2000): Mellin transforms of Hermite and Laguerre functions form families of orthogonal polynomials and have zeros on the critical line $\Re(s) = 1/2$. Coffey (2007), Coffey and Lettington (2015) consider other families.

Moments of Gaussian/Wishart random matrices can be represented as Mellin transforms of Wronskians of consecutive Hermite/Laguerre functions:

$$\mathbb{E}\,\mathrm{tr}\,X_n^{-s} = c(s) \int x^{-s}\,\mathrm{Wr}\,(\phi_{n-1}, \phi_n)(x)\,dx.$$

cf. exceptional orthogonal polynomials, etc.

Zeros of Bessel functions

Polya (1926) considered an approximation of Riemann's zeta function for which the Riemann hypothesis is implied by the statement that the zeros of $z \mapsto K_z(2\pi)$ are purely imaginary.

This fact (and spectral interpretation of the zeros) can be proved using Sturm-Lioville theory (see, for e.g., Biane 2009): consider

$$\left(\frac{d^2}{dx^2} - e^{2x}\right) K_\mu(e^x) = \mu^2 K_\mu(e^x)$$

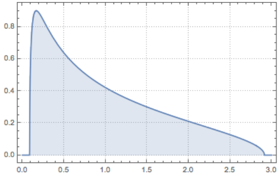
with Dirichlet boundary conditions on $[\log(2\pi), \infty)$.

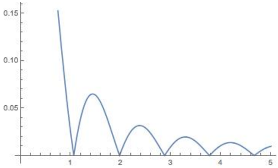
Mellin transform of Marchenko-Pastur distribution

We find that the Mellin transform of the Marchenko-Pastur distribution

$$M(s) = \frac{1}{2\pi}\int x^{-s-1}\sqrt{(x-a_-)(a_+-x)}dx, \qquad a_\pm = (1\pm\sqrt{2})^2$$

has a similar interpretation, satisfies the functional equation $M(s) = M(1-s)$, and its zeros lie on the critical line $\Re(s) = 1/2$.





Orthogonal and symplectic ensembles

The above examples are over $\mathbb{C}$, but can also be considered over $\mathbb{R}$ and $\mathbb{H}$.

Writing $Q_k^{\mathbb{K}}(n) = \mathbb{E}\operatorname{tr} X^{2k}$, we have for example in the Gaussian case:

$$
\begin{aligned}
Q_2^{\mathbb{C}}(n) &= 2n^3 + n \\
Q_2^{\mathbb{R}}(n) &= 2n^3 + 5n^2 + 5n \\
Q_2^{\mathbb{H}}(n) &= 8n^3 - 10n^2 + 5n.
\end{aligned}
$$

The real case is a genus expansion which includes non-orientable surfaces.

Reference: Goulden-Jackson (1996)

Orthogonal and symplectic ensembles

We find that

$$p_n(k) = c_n 2^{k+1} Q_k^{\mathbb{H}}(n)/(2k-1)!!$$

is a polynomial in k of degree $2(n-1)$:

$$\begin{aligned}
p_1(k) &= 1 \\
p_2(k) &= k^2 + 5k + 3 \\
p_3(k) &= k^4 + 10k^3 + 38k^2 + 41k + \frac{45}{2} \\
p_4(k) &= k^6 + 15k^5 + 109k^4 + 393k^3 + 637k^2 + 735k + 315.
\end{aligned}$$

Orthogonal and symplectic ensembles

For the real case we find that

$$c_n Q_k^{\mathbb{R}}(2n+1)/(2k-1)!! = p_n(k) + 2^k q_n(k)$$

where $q_n(k)$ is a Meixner polynomial of degree n.

Remark: this is a new duality formula, compare

$$Q_k^{\mathbb{R}}(2n+1) = 2^{k+1} Q_k^{\mathbb{H}}(n) + 2^k q_n(k)/c_n$$

with the known formula (Mulase-Waldron 2003)

$$Q_k^{\mathbb{H}}(n) = (-1)^{k+1} Q_k^{\mathbb{R}}(-2n).$$

Polya's 'false' zeta function

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \qquad \xi(s) = \xi(1-s)$$

$$\xi(1/2+iz) = 2\int_0^\infty \phi(u)\cos(zu)du$$

$$\begin{aligned}\phi(u) &= 2\pi e^{5u/2}\sum_{n=1}^{\infty}(2\pi e^{2u}n^2-3)n^2 e^{-\pi n^2 e^{2u}} \\ &\sim 4\pi^2(e^{9u/2}+e^{-9u/2})e^{-\pi(e^{2u}+e^{-2u})} \qquad u\to\pm\infty.\end{aligned}$$

Polya (1926) defines

$$\begin{aligned}\xi^*(z) &= 2\int_0^\infty 4\pi^2(e^{9u/2}+e^{-9u/2})e^{-\pi(e^{2u}+e^{-2u})}\cos(zu)du \\ &= 2\pi^2\left[K_{iz/2-9/4}(2\pi)+K_{iz/2+9/4}(2\pi)\right].\end{aligned}$$

Kac (1974): if $K_{iz}(2\pi)$ has real zeros, then so does $\xi^*(z)$, by the Lee-Yang theorem.