From longest increasing subsequences to Whittaker functions and random polymers

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The longest increasing subsequence problem

For a permutation $\sigma \in S_n$, write

$$L_n(\sigma) = \text{ length of longest increasing subsequence in } \sigma$$

E.g. if $\sigma = 154263$ then $L_6(\sigma) = 3$.

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Based on Monte-Carlo simulations, Ulam (1961) conjectured that

$$EL_n = \frac{1}{n!} \sum_{\sigma \in S_n} L_n(\sigma) \sim c\sqrt{n}, \qquad n \to \infty.$$

A classical result from combinatorial geometry (Erdős-Szekeres 1935) implies that $EL_n \geq \sqrt{n-1}/2$.

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Baik, Deift and Johansson (1999): for each $x \in \mathbb{R}$,

$$\frac{1}{n!}|\{\sigma \in S_n : \ n^{-1/6}(L_n(\sigma) - 2\sqrt{n}) \leq x\}| \to F_2(x),$$

where F_2 is the Tracy-Widom (GUE) distribution from random matrix theory (Tracy and Widom 1994 — limiting distribution of largest eigenvalue of high-dimensional random Hermitian matrix)

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How is this possible?

The Robinson-Schensted correspondence

From the representation theory of S_n,

$$n! = \sum_{\lambda \vdash n} d_\lambda^2$$

where d_λ = number of standard tableaux with shape λ.

A standard tableau with shape $(4, 3, 1) \vdash 8$:

1	3	5	6
2	4	8	
7			

In other words, S_n has the same cardinality as the set of pairs of standard tableaux of size n with the same shape.

The Robinson-Schensted correspondence

Robinson (38): A bijection between S_n and such pairs

$$\sigma \longleftrightarrow (P, Q)$$

Schensted (61):

$$L_n(\sigma) = \text{ length of longest row of } P \text{ and } Q$$

This yields

$$|\{\sigma \in S_n : L_n(\sigma) \leq k\}| = \sum_{\lambda \vdash n,\ \lambda_1 \leq k} d_\lambda^2.$$

The RSK correspondence

Knuth (70): Extends to a bijection between matrices with nonnegative integer entries and pairs of *semi-standard* tableaux of same shape.

A *semistandard tableau* of shape $\lambda \vdash n$ is a diagram of that shape, filled in with positive integers which are *weakly* increasing along rows and strictly increasing along columns.

A semistandard tableau of shape $(5, 3, 1)$:

1	2	2	5	7
3	3	8		
4				

Cauchy-Littlewood identity

This gives a combinatorial proof of the Cauchy-Littlewood identity

$$\prod_{ij}(1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(x) s_\lambda(y),$$

where s_λ are Schur polynomials, defined by

$$s_\lambda(x) = \sum_{\text{sh } P=\lambda} x^P,$$

where $x = (x_1, x_2, \dots)$ and

$$x^P = x_1^{\sharp 1's \ in \ P} x_2^{\sharp 2's \ in \ P} \dots .$$

Cauchy-Littlewood identity

Let $(a_{ij}) \mapsto (P, Q)$ under RSK.

Then $C_j = \sum_i a_{ij} = \sharp\, j$'s in P and $R_i = \sum_j a_{ij} = \sharp\, i$'s in Q.

For $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ we have

$$\prod_{ij} (y_i x_j)^{a_{ij}} = \prod_j x_j^{C_j} \prod_i y_i^{R_i} = x^P y^Q.$$

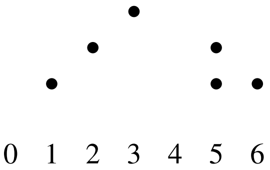
Summing over (a_{ij}) on the left and (P, Q) with sh $P =$ sh Q on the right gives

$$\prod_{ij} (1 - x_i y_j)^{-1} = \sum_\lambda s_\lambda(x) s_\lambda(y).$$

Tableaux and Gelfand-Tsetlin patterns

Semistandard tableaux $\longleftrightarrow$ discrete Gelfand-Tsetlin patterns

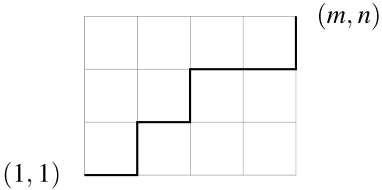
1	1	1	2	2	3
2	2	3	3	3	
3					



The RSK correspondence

If $(a_{ij}) \in \mathbb{N}^{m\times n}$, then length of longest row in corresponding tableaux is

$$M = \max_\pi \sum_{(i,j)\in\pi} a_{ij}$$



Combinatorial interpretation

Consider n queues in series:

☺ ☺ ☺| ⟶ ☹ ☺| ⟶ ☹ ☹ ☺| ⟶ .... ☺| ⟶

Data:

$$a_{ij} = \text{ time required to serve } i^{th} \text{ customer at } j^{th} \text{ queue}$$

If we start with all customers in first queue, then M is the time taken for all customers to leave the system (Muth 79).

Combinatorial interpretation

From the RSK correspondence:

If a_{ij} are independent random variables with $P(a_{ij} \geq k) = (p_i q_j)^k$ then

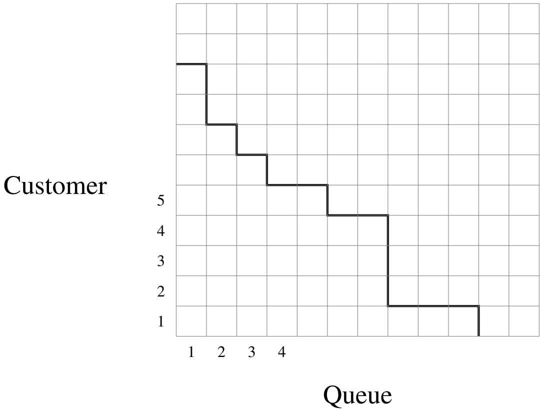
$$P(M \leq k) = \prod_{ij}(1 - p_i q_j) \sum_{\lambda:\ \lambda_1 \leq k} s_\lambda(p) s_\lambda(q).$$

cf. Weber (79): *The interchangeability of* $\cdot/M/1$ *queues in series.*

Johansson (99): As $n, m \to \infty$, $M \sim$ Tracy-Widom distribution
(and other related asymptotic results)

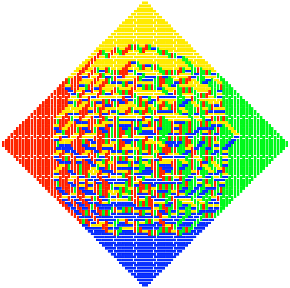
Surface growth and KPZ universality

The queueing system can be thought of as a model for surface growth . . .



Surface growth and KPZ universality

…and belongs to the same *universality class* as:



Random tiling

Burning paper

Bacteria colonies

KPZ = Kardar-Parisi-Zhang (1986)

Geometric RSK correspondence

The RSK mapping can be defined by expressions in the $(\max,+)$-semiring.

Replacing these expressions by their $(+,\times)$ counterparts, A.N. Kirillov (00) introduced a *geometric lifting* of RSK correspondence.

It is a bi-rational map

$$T : (\mathbb{R}_{>0})^{n\times n} \to (\mathbb{R}_{>0})^{n\times n}$$

$$X = (x_{ij}) \mapsto (t_{ij}) = T = T(X).$$

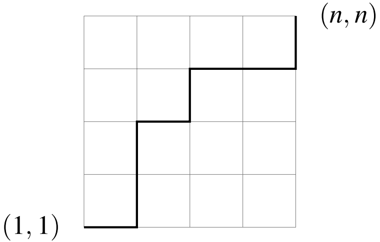
For $n = 2$,

$$\begin{matrix} & x_{21} & \\ x_{11} & & x_{22} \\ & x_{12} & \end{matrix} \quad \mapsto \quad \begin{matrix} & x_{11}x_{21} & \\ x_{12}x_{21}/(x_{12}+x_{21}) & & x_{11}x_{22}(x_{12}+x_{21}) \\ & x_{11}x_{12} & \end{matrix}$$

Geometric RSK correspondence

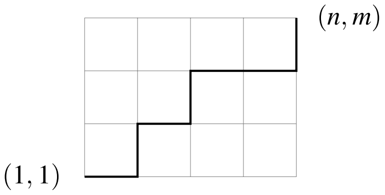
The analogue of the 'longest increasing subsequence' is the matrix element:

$$t_{nn} = \sum_{\phi \in \Pi_{(n,n)}} \prod_{(i,j)\in\phi} x_{ij}$$



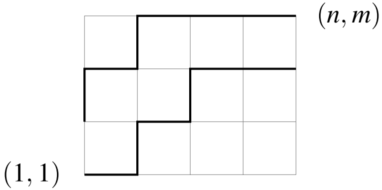
Geometric RSK correspondence

$$t_{nm} = \sum_{\phi \in \Pi_{(n,m)}} \prod_{(i,j) \in \phi} x_{ij}$$



Geometric RSK correspondence

$$t_{n-k+1,m-k+1} \dots t_{nm} = \sum_{\phi \in \Pi^{(k)}_{(n,m)}} \prod_{(i,j) \in \phi} x_{ij}$$



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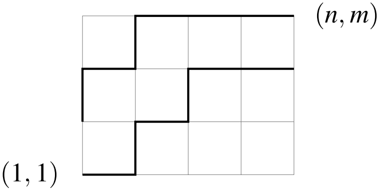
$$t_{n-k+1,m-k+1} \dots t_{nm} = \sum_{\phi \in \Pi^{(k)}_{(n,m)}} \prod_{(i,j)\in\phi} x_{ij}$$

$$T(X)' = T(X')$$



Whittaker functions

- Whittaker functions were first introduced by Jacquet (67). They play an important role in the theory of automorphic forms and also arise as eigenfunctions of the open quantum Toda chain (Kostant 77)

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- In the context of $GL(n,\mathbb{R})$, they can be considered as functions $\Psi_\lambda(x)$ on $(\mathbb{R}_{>0})^n$, indexed by a (spectral) parameter $\lambda \in \mathbb{C}^n$
- The following 'Gauss-Givental' representation for Ψ_λ is due to Givental (97), Joe-Kim (03), Gerasimov-Kharchev-Lebedev-Oblezin (06)

Whittaker functions

A *triangle* P with shape $x \in (\mathbb{R}_{>0})^n$ is an array of positive real numbers:

$$P = \begin{array}{ccccccc} & & & z_{11} & & & \\ & & z_{22} & & z_{21} & & \\ & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\ z_{nn} & & & \cdots\cdots & & & z_{n1} \end{array}$$

with bottom row $z_{n\cdot} = x$.

Denote by $\Delta(x)$ the set of triangles with shape x.

Whittaker functions

Let

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Define

$$P^\lambda = R_1^{\lambda_1} \left(\frac{R_2}{R_1} \right)^{\lambda_2} \cdots \left(\frac{R_n}{R_{n-1}} \right)^{\lambda_n}, \qquad \lambda \in \mathbb{C}^n, \qquad R_k = \prod_{i=1}^{k} z_{ki}$$

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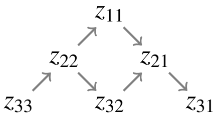
$$\mathcal{F}(P) = \sum_{a \to b} \frac{z_a}{z_b}$$



Whittaker functions

For $\lambda \in \mathbb{C}^n$ and $x \in (\mathbb{R}_{>0})^n$, define

$$\Psi_\lambda(x) = \int_{\Delta(x)} P^{-\lambda} e^{-\mathcal{F}(P)} dP,$$

where $dP = \prod_{1 \le i \le k < n} dz_{ki}/z_{ki}$.

For $n = 2$,

$$\Psi_{(\nu/2,-\nu/2)}(x) = 2K_\nu\left(2\sqrt{x_2/x_1}\right).$$

These are called $GL(n)$-Whittaker functions. As we shall see, they are the analogue of the Schur polynomials in the geometric setting.

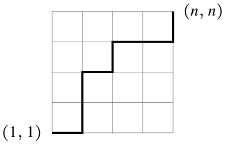
Geometric RSK correspondence

Recall

$$X = (x_{ij}) \mapsto (t_{ij}) = T \quad = \quad \begin{array}{ccccc} & & t_{31} & & \\ & t_{21} & & t_{32} & \\ t_{11} & & t_{22} & & t_{33} \\ & t_{12} & & t_{23} & \\ & & t_{13} & & \end{array}$$

= pair of triangles of same shape $(t_{nn}, \ldots, t_{11})$.

$$t_{nn} = \sum_{\phi \in \Pi_{(n,n)}} \prod_{(i,j)\in\phi} x_{ij}$$



Whittaker measures

Let $a, b \in \mathbb{R}^n$ with $a_i + b_j > 0$ and define

$$\mathbb{P}(dX) = \prod_{ij} \Gamma(a_i + b_j)^{-1} x_{ij}^{-a_i - b_j - 1} e^{-1/x_{ij}} dx_{ij}.$$

Theorem (Corwin-O'C-Seppäläinen-Zygouras 14)

Under $\mathbb{P}$, the law of the shape of the output under geometric RSK is given by the Whittaker measure *on $\mathbb{R}_+^n$ defined by*

$$\mu_{a,b}(dx) = \prod_{ij} \Gamma(a_i + b_j)^{-1} e^{-1/x_n} \Psi_a(x) \Psi_b(x) \prod_{i=1}^{n} \frac{dx_i}{x_i}.$$

Application to random polymers

Corollary

Suppose $a_i > 0$ for each i and $b_j < 0$ for each j. Then

$$\mathbb{E}e^{-st_{nn}} = \int_{\iota\mathbb{R}^m} s^{\sum_{i=1}^n (b_i-\lambda_i)} \prod_{ij} \Gamma(\lambda_i - b_j) \prod_{ij} \frac{\Gamma(a_i+\lambda_j)}{\Gamma(a_i+b_j)} s_n(\lambda) d\lambda,$$

where

$$s_n(\lambda) = \frac{1}{(2\pi\iota)^n n!} \prod_{i\neq j} \Gamma(\lambda_i - \lambda_j)^{-1}.$$

Combinatorial approach

Recall: $X = (x_{ij}) \mapsto (t_{ij}) = T(X) = (P, Q)$.

The following is a refinement of the previous theorem.

Theorem (O'C-Seppäläinen-Zygouras 14)

- *The map* $(\log x_{ij}) \to (\log t_{ij})$ *has Jacobian* ± 1
- *For* $\nu, \lambda \in \mathbb{C}^n$,
$$\prod_{ij} x_{ij}^{\nu_i + \lambda_j} = P^\lambda Q^\nu$$
- *The following identity holds:*
$$\sum_{ij} \frac{1}{x_{ij}} = \frac{1}{t_{11}} + \mathcal{F}(P) + \mathcal{F}(Q)$$

This theorem (a) 'explains' the appearance of Whittaker functions and (b) extends to models with symmetry.

Analogue of the Cauchy-Littlewood identity

It follows that

$$\prod_{ij} x_{ij}^{-\nu_i-\lambda_j} e^{-1/x_{ij}} \frac{dx_{ij}}{x_{ij}} = P^{-\lambda} Q^{-\nu} e^{-1/t_{11}-\mathcal{F}(P)-\mathcal{F}(Q)} \prod_{ij} \frac{dt_{ij}}{t_{ij}}.$$

Integrating both sides gives, for $\Re(\nu_i + \lambda_j) > 0$:

Corollary (Stade 02)

$$\prod_{ij} \Gamma(\nu_i + \lambda_j) = \int_{\mathbb{R}_+^n} e^{-1/x_n} \Psi_\nu(x) \Psi_\lambda(x) \prod_{i=1}^n \frac{dx_i}{x_i}.$$

This is equivalent to a Whittaker integral identity which was conjectured by Bump (89) and proved by Stade (02). The integral is associated with Archimedean L-factors of automorphic L-functions on $GL(n, \mathbb{R}) \times GL(n, \mathbb{R})$.

Local moves

Proof of second theorem uses new description of the gRSK map T as a composition of a sequence of 'local moves' applied to the input matrix

$$
\begin{array}{ccccc}
 & & x_{31} & & \\
 & x_{21} & & x_{32} & \\
x_{11} & & x_{22} & & x_{33} \\
 & x_{12} & & x_{23} & \\
 & & x_{13} & &
\end{array}
$$

Local moves

The basic move is:

$$b$$

$$a \qquad\qquad d$$

$$c$$

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$$b$$

$$\frac{bc}{ab+ac} \qquad bd+cd$$

$$c$$

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Integrable systems point of view

Quantum Toda chain

$$H = \Delta - 2 \sum_{i=1}^{n-1} e^{x_{i+1} - x_i}$$

$$H\psi_\lambda = \Big(\sum_i \lambda_i^2\Big)\psi_\lambda \qquad \psi_\lambda(x) = \Psi_{-\lambda}(e^x)$$

Associated diffusion process in $\mathbb{R}^n$ with generator

$$L_\lambda = \tfrac{1}{2}\psi_\lambda(x)^{-1}\Big(H - \sum \lambda_i^2\Big)\psi_\lambda(x) = \tfrac{1}{2}\Delta + \nabla \log \psi_\lambda \cdot \nabla.$$

Geometric RSK (in continuous time)

Let $\eta : [0,\infty) \to \mathbb{R}^n$ be smooth (or Brownian) with $\eta(0) = 0$.

Define $b(t)$ in upper triangular matrices by

$$\dot{b} = \epsilon(\dot{\eta})b, \qquad b(0) = Id.,$$

where

$$\epsilon(\lambda) = \begin{pmatrix} \lambda_1 & 1 & 0 & \dots & 0 \\ 0 & \lambda_2 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ & & & \lambda_{n-1} & 1 \\ 0 & & \dots & & \lambda_n \end{pmatrix}.$$

Consider the 'principal' minors

$$\Delta_k = \det \left[b_{ij} \right]_{1 \le i \le k,\ n-k+1 \le j \le n}, \qquad 1 \le k \le n$$

and define $x_i = \log(\Delta_i / \Delta_{i-1})$, where $\Delta_0 = 1$.

Geometric RSK (in continuous time)

The main results described above are in fact discrete-time versions of:

Theorem (O'C 12)

If $\eta(t) = B(t) + \lambda t$ then $x(t)$ is a diffusion with generator L_λ.

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What happens if we remove the 'noise'?

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Theorem (O'C 13)

If $\eta(t) = \lambda t$ then $x(t)$ is a solution to the classical Toda flow with constants of motion (Lax matrix eigenvalues) $\lambda_1, \dots, \lambda_n$.

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This phenomena extends to hyperbolic Calogero-Moser systems. In all cases, the quantum systems (in imaginary time) are obtained by adding noise to the constants of motion in particular representations of the classical systems.

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