From Pitman's $2M - X$ theorem to random polymers and integrable systems

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Acknowledgements

Collaborators

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References

Neil O'Connell and Marc Yor. A representation for non-colliding random walks. *Electron. Comm. Probab.* 7 (2002).

Philippe Biane, Philippe Bougerol and Neil O'Connell. Littelmann paths and Brownian paths. *Duke Math. J.* 130 (2005) 127–167.

Philippe Biane, Philippe Bougerol and Neil O'Connell. Continuous crystals and Duistermaat-Heckman measure for Coxeter groups. *Adv. Math.* 221 (2009) 1522-1583.

Neil O'Connell. Directed polymers and the quantum Toda lattice. *Ann. Probab.* 40 (2012) 437-458.

Pitman's $2M - X$ theorem (1975)

If $(X(t), t \geq 0)$ is a one-dimensional Brownian motion and

$$M(t) = \max_{s \leq t} B(s)$$

then

$$R(t) = 2M(t) - X(t)$$

is a three-dimensional Bessel process.

Pitman's $2M - X$ theorem (1975)

Equivalently, if $(B(t), t \geq 0)$ is a one-dimensional Brownian motion, then

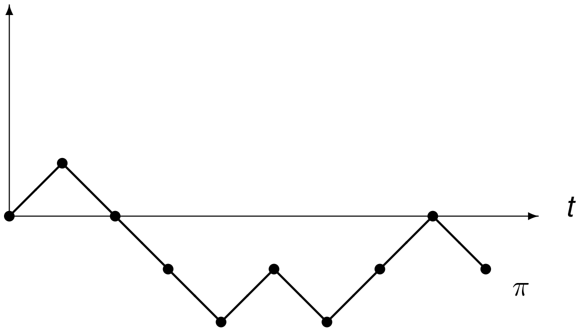
$$B(t) - 2 \inf_{s \leq t} B(s), \qquad t \geq 0$$

is a three-dimensional Bessel process.

The Pitman transform

For continuous $\pi : [0, \infty) \to \mathbb{R}$ with $\pi(0) = 0$, define $P\pi$ by

$$P\pi(t) = \pi(t) - 2 \inf_{s \le t} \pi(s).$$



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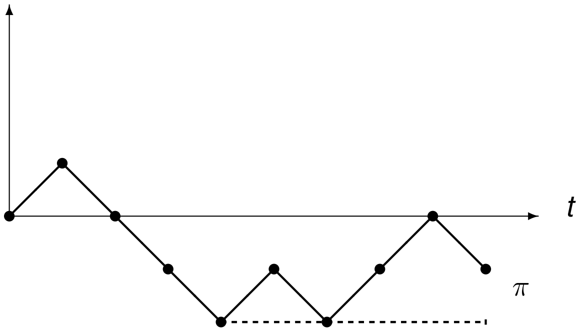

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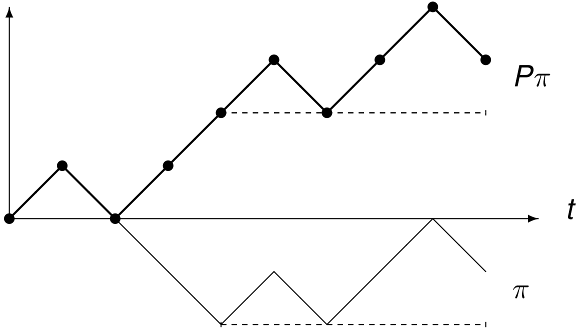

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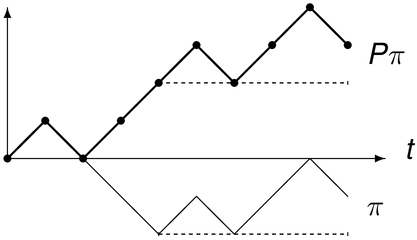
Note that $P^2 = P$.



Generalised Pitman Transforms

Biane-Bougerol-O'C 05

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Let V be a finite-dimensional Euclidean space.

For $\alpha \in V$, set $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

For continuous $\eta : [0, \infty) \to V$ and $\alpha \in V$, define

$$P_\alpha \eta(t) = \eta(t) - \inf_{s \leq t} \alpha^\vee(\eta(s))\alpha.$$

E.g. if $V = \mathbb{R}$ and $\alpha = 1$, then

$$P_\alpha \eta(t) = \eta(t) - 2 \inf_{s \leq t} \eta(s).$$

Hyperplane reflections and braid relations

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Let $\beta \in V$ with $(\alpha, \beta) = -|\alpha|\,|\beta| \cos(\pi/n)$. Then

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Theorem (Biane, Bougerol, O'C 05)

$$P_\alpha P_\beta P_\alpha \cdots = P_\beta P_\alpha P_\beta \cdots \qquad n \textit{ terms}$$

Finite Coxeter groups

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If $\pi/n_{\alpha\beta}$ is the angle between hyperplanes $\alpha^\perp$ and $\beta^\perp$ then

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The (closure of)

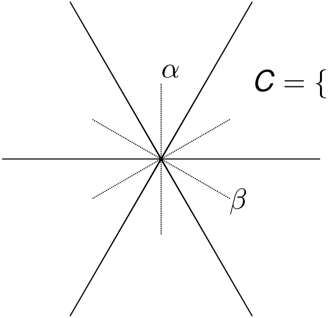
$$C = \{\lambda \in V : \ (\alpha, \lambda) > 0, \ \forall \alpha \in \Delta\}$$

is a fundamental domain for the action of W on V.

Example

$$V = \mathbb{R}^3 \qquad \alpha = e_1 - e_2 \qquad \beta = e_2 - e_3$$

$$W = \langle\, s_\alpha,\, s_\beta \,\rangle \simeq S_3 = \langle\, (12),\, (23) \,\rangle$$



$$C = \{x_1 > x_2 > x_3\}$$

Generalised Pitman Transforms II

Corollary (Biane-Bougerol-O'C 05)

For each $w \in W$, we can define

$$P_w = P_{\alpha_1} \cdots P_{\alpha_k}$$

where $w = s_{\alpha_1} \cdots s_{\alpha_k}$ is any *reduced decomposition of w.*

The longest element

Let W be a finite Coxeter group with generating simple reflections $S = \{s_\alpha, \alpha \in \Delta\}$. The length of an element $w \in W$ is the minimal number of terms required to write w as a product of simple reflections. There is a unique $w_0 \in W$ of maximal length.

For example, the longest element in S_3 is

$$(13) = (12)(23)(12) = (23)(12)(23).$$

The longest element in S_n is

$$w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}.$$

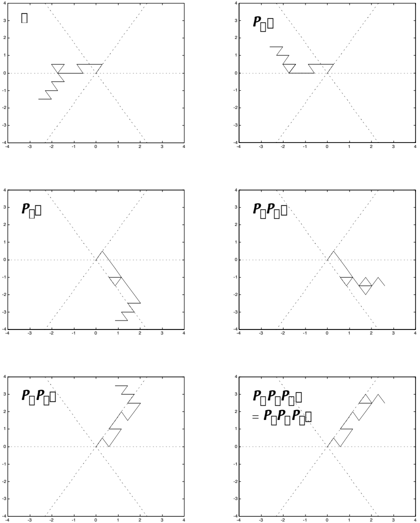
Some properties of P_{w_0}

Theorem (Biane-Bougerol-O'C 05)

For any continuous $\eta : [0, \infty) \to V$ *with* $\eta(0) = 0$, $P_{w_0}\eta(t) \in \overline{C}$ *for all* $t \geq 0$. *Furthermore,*

$$P_{w_0}^2 = P_{w_0}$$

First claim is a consequence of the braid relations.



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Biane (1993): There exists a unique (up to constant factors) positive p^*-harmonic function h on C.

Brownian motion (conditioned to stay) in a cone

Let C be a convex cone in V.

Let $p^*_t(x,y)$ be the transition density of Brownian motion killed when it exits C.

Biane (1993): There exists a unique (up to constant factors) positive p^*-harmonic function h on C.

Brownian motion in the cone C is the corresponding Doob h-transform, with generator and transition density

$$\frac{1}{2}\Delta + \nabla(\log h)\cdot\nabla \qquad\qquad q_t(x,y) = \frac{h(y)}{h(x)} p^*_t(x,y).$$

The three-dimensional Bessel Process

If $V = \mathbb{R}$ and $C = \mathbb{R}_+$ then

$$p_t^*(x,y) = p_t(x,y) - p_t(x,-y) \qquad h(x) = x.$$

Brownian motion in $\mathbb{R}_+$ is the three-dimensional Bessel process, with infinitessimal generator

$$\frac{1}{2}\frac{d}{dx^2} + \frac{1}{x}\frac{d}{dx}.$$

Brownian motion in a Weyl chamber

Let W be a finite Coxeter group acting on V with fundamental chamber C. Then

$$p_t^*(x,y) = \sum_{w \in W} \varepsilon(w) p_t(x, wy) \qquad h(x) = \prod_{\alpha \in \Phi^+} \alpha^\vee(x).$$

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If $W = S_n$ and $V = \mathbb{R}^n$, then

$$C = \{x \in \mathbb{R}^n : \ x_1 > x_2 > \cdots > x_n\} \qquad h(x) = \prod_{i<j} (x_i - x_j)$$

and the Brownian motion in C has the same law as the eigenvalue process of a Brownian motion in the space of $n \times n$ Hermitian matrices (aka Dyson's Brownian motion).

A generalisation of Pitman's theorem

Let W be a finite Coxeter group acting on V with fundamental chamber C.

Theorem (Biane-Bougerol-O'C 05)

If η is a Brownian motion in V then $P_{w_0}\eta$ has the same law as a Brownian motion in C.

Proof uses queueing theoretic ideas and algebraic properties of the Pitman operators.

Result for $W = S_n$ case - O'C-Yor 02, Bougerol-Jeulin 02

For $W = S_2$, reduces to Pitman's $2M - X$ theorem.

A concrete application

Brownian TASEP / ‘heavy traffic’ queues in series
Harrison 73, Harrison-Williams 87, Glynn-Whitt 91

Independent Brownian motions on $\mathbb{R}$, each particle reflected off particle its left, all particles started at the origin, denote by $X^n(t)$ the position of the n^{th} particle at time t.

Corollary (O’C-Yor 02, Bougerol-Jeulin 02)

The process $X^n(t)$ has the same law as the largest eigenvalue of an $n \times n$ Hermitian Brownian motion started from zero.

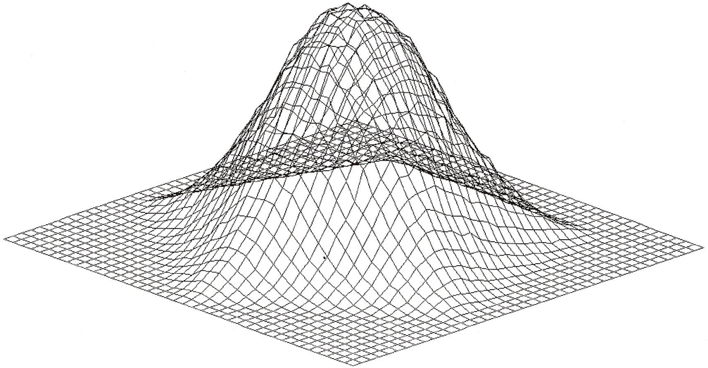
Corollary (Baryshnikov 01, Gravner-Tracy-Widom 01)

The random variable $X^n(1)$ has the same law as the largest eigenvalue of an $n \times n$ GUE random matrix.

A more esoteric application

Duistermaat-Heckman measure for finite Coxeter groups
Biane-Bougerol-O'C 09

E.g. Here is a natural measure on the pentagon:



Exponential version of Pitman's $2M - X$ theorem

Theorem (Matsumoto-Yor '99)

Let $(B_t, t \geq 0)$ *be a standard one-dimensional Brownian motion and define*

$$Z_t = \int_0^t e^{2B_s - B_t} ds.$$

Then $\log Z_t$ *is a diffusion with infinitesimal generator*

$$\frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{d}{dx}\log K_0(e^{-x})\right)\frac{d}{dx}.$$

The above diffusion can be interpreted as BM conditioned, in the sense of Doob, to survive in the potential e^{-x}.

Generalised Matsumoto-Yor transforms

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For $\eta : (0,\infty) \to V$ and $\alpha \in V$, define

$$T_\alpha \eta(t) = \eta(t) + \left(\log \int_0^t e^{-\alpha(\eta(s))} ds \right) \alpha^\vee .$$

Braid relations

Theorem [Biane, Bougerol, O'C 05]

Let $\alpha, \beta \in V$ with $(\alpha, \beta) = -\cos(\pi/n)$, where $n \in \{2, 3, 4, 6\}$. Then

$$T_\alpha T_\beta T_\alpha \cdots = T_\beta T_\alpha T_\beta \cdots \qquad n \text{ terms}$$

Thus, if W is a crystallographic finite reflection group acting on V with simple reflections $S = \{s_\alpha, \alpha \in \Delta\}$, then for each $w \in W$, we can define

$$T_w = T_{\alpha_1} \cdots T_{\alpha_k}$$

where $w = s_{\alpha_1} \cdots s_{\alpha_k}$ is *any* reduced decomposition of w.

The quantum Toda lattice

The quantum Toda lattice is a quantum integrable system with Hamiltonian given by

$$H = \Delta - 2\sum_{i=1}^{n-1} e^{x_{i+1}-x_i}.$$

There is a (particular) positive eigenfunction ψ_0 with $H\psi_0 = 0$ (known as a class one Whittaker function)

When $n = 2$,

$$\psi_0(x) = 2K_0\left(2e^{(x_2-x_1)/2}\right).$$

Multi-dimensional version of Matsumoto-Yor theorem

Let $V = \mathbb{R}^n$ and $W = S_n$.

Theorem (O'C 12)

If η a standard Brownian motion in $\mathbb{R}^n$ then $T_{w_0}\eta$ is a diffusion in $\mathbb{R}^n$ with generator given by the Doob transform

$$\frac{1}{2}\psi_0^{-1}H\psi_0 = \frac{1}{2}\Delta + \nabla \log \psi_0 \cdot \nabla.$$

This diffusion can be interpreted as BM($\mathbb{R}^n$) conditioned, in the sense of Doob, to survive in the potential $\sum_{i=1}^{n-1} e^{x_{i+1}-x_i}$ (see Katori 2011). It was introduced in [Baudoin-O'C 2011] and is the analogue of Dyson's BM in this setting.

A random polymer model

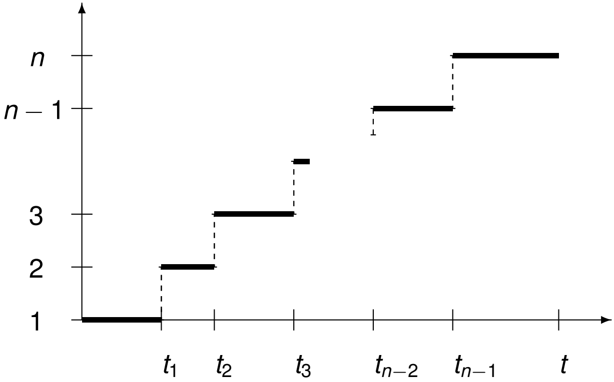
$B_1, B_2, \ldots$ be independent standard Brownian motions,

$$Z_t^n = \int_{0=t_0<t_1<\cdots<t_{n-1}<t_n=t} e^{\sum_{i=1}^n B_i(t_i)-B_i(t_{i-1})} dt_1 \ldots dt_{n-1}.$$

This is the partition function of a $1+1$ dim. directed polymer in a random environment (O'C-Yor 01, O'C-Moriarty 07).

If $\eta = (B_n, \ldots, B_1)$ then $\log Z_t^n = (T_{w_0}\eta)_1(t)$. The above theorem thus determines the law of the partition function for this model and moreover provides explicit determinantal formulae.

A random polymer model



A polymer realisation $\{0 < t_1 < \ldots < t_{n-1} < t\}$

Further and related developments

Extensions to other root systems *Chhaibi 2012*

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Discrete time versions in context of geometric RSK correspondence, with applications to lattice polymers with log-gamma weights and various symmetries; Whittaker functions also play a central role in this setting
Corwin-O'C-Seppalainen-Zygouras 2011,
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Determinantal formulae for law of solution of KPZ equation (continuum random polymer)
Amir-Corwin-Quastel 2010, Sasamoto-Spohn 2010
Calabrese-Le Doussal-Rosso 2010, Dotsenko 2010