Directed polymers and the quantum Toda lattice

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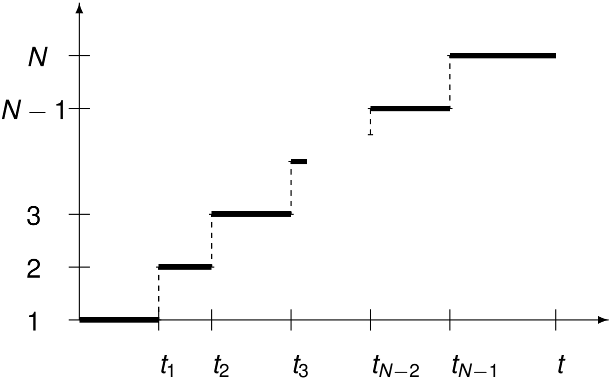
A directed polymer model



A path $\phi \equiv \{0 < t_1 < \ldots < t_{N-1} < t\}$.

A directed polymer model

The environment: $B_1, B_2, \ldots$ independent standard 1-dim Brownian motions.

For $\phi \equiv \{0 < t_1 < \ldots < t_{N-1} < t\}$, define

$$E(\phi) = B_1(t_1) + B_2(t_2) - B_2(t_1) + \cdots + B_N(t) - B_N(t_{N-1}).$$

Boltzmann measure:

$$\mathbb{P}(d\phi) = Z_t^N(\beta)^{-1} e^{\beta E(\phi)} d\phi, \qquad Z_t^N(\beta) = \int e^{\beta E(\phi)} d\phi.$$

The free energy density

Theorem (O'C-Yor '01, O'C-Moriarty '07)

Almost surely,

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N^N(\beta) = \inf_{t>0}[t\beta^2 - \Psi(t)] - \log \beta^2 =: f(\beta),$$

where $\Psi(z) = \Gamma'(z)/\Gamma(z)$.

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For small β,

$$\lim_{N\to\infty} \frac{1}{N} \log \frac{Z_N^N(\beta)}{\mathbb{E} Z_N^N(\beta)} \sim \frac{5}{24}\beta^4.$$

Fluctuations

Theorem (Seppalainen-Valko 2010)

There exist finite, positive β-dependent constants C, b_0 and N_0 such that for $b \geq b_0$ and $N \geq N_0$,

$$\mathbb{P}\left(|\log Z_N^N(\beta) - f(\beta)N| \geq bN^{1/3} \right) \leq Cb^{-3/2}.$$

A scaling property

By Brownian scaling,

$$(Z_t^N(\beta), t \geq 0) \stackrel{d}{=} (\beta^{-2(N-1)} Z_{\beta^2 t}^N(1), t \geq 0).$$

Define

$$Z_t^N = Z_t^N(1).$$

As an interacting particle system ...

Set $X_t^N = \log Z_t^N$. Then

$$dX_t^N = e^{X_t^{N-1} - X_t^N} dt + dB_t^N.$$

Infinite system has product-form invariant measure for each given intensity.

This allows computation of the free energy density following Rost (1986) / Seppalainen (1998), analogous to TASEP.

Connection with random matrices

The law of $Z_t^N(\beta)$ is well-understood in the zero temperature limit. Define

$$\begin{aligned} M_t^N &= \lim_{\beta\to\infty} \frac{1}{\beta} \log Z_t^N(\beta) \\ &= \max_{0=t_0\le t_1\le\cdots\le t_{N-1}\le t_n=t} \sum_{i=1}^N B_i(t_i) - B_i(t_{i-1}). \end{aligned}$$

The process $(M_t^N,\ t\ge 0)$ is B^N 'reflected off' B^{N-1} 'reflected off' ... 'reflected off' B^2 'reflected off' B^1.

By Brownian scaling, the law of $M_t^N/\sqrt{t}$ is independent of t.

Connection with random matrices

Theorem (Baryshnikov '01, Gravner-Tracy-Widom '01)

The random variable M_1^N has the same law as the largest eigenvalue of a $N \times N$ GUE random matrix, that is

$$\mathbb{P}(M_1^N \leq y) = \int_{\max_{1\leq i\leq N} x_i \leq y} c_N e^{-\sum_{i=1}^N x_i^2/2} h(x)^2 dx$$

where

$$h(x) = \prod_{1\leq i<j\leq N} (x_i - x_j)$$

and c_N is a normalisation constant.

Connection with random matrices

This yields very precise information about the law and asymptotic behavior of M^N. For example,

$$P(M_N^N \leq y) = \det[1 - K_N]_{L_2([y,\infty)}$$

where K_N is the ‘Hermite kernel’, and

$$\lim_{N\to\infty} P\left(M_N^N \leq 2N + yN^{1/3}\right) = F_2(y),$$

where

$$F_2(y) = \det[1 - K_{Airy}]_{L_2([y,\infty)}$$

is the Tracy-Widom distribution.

Connection with random matrices

In fact [Bougerol-Jeulin '02, O'C-Yor '02] the stochastic process $(M_t^N,\ t \geq 0)$ has the same law as the top line of a system of N Dyson Brownian motions. That is, it has the same law as the first coordinate of a Brownian motion conditioned never to exit

$$C_N = \{x \in \mathbb{R}^N :\ x_1 > \cdots > x_N\},$$

started from the origin. This is a diffusion in $\overline{C}_N$ with generator

$$\frac{1}{2} h(x)^{-1} \Delta_{C_N} h(x) = \Delta/2 + \nabla \log h \cdot \nabla.$$

The quantum Toda lattice

The quantum Toda lattice is a quantum integrable system with Hamiltonian given by

$$H = \Delta - 2 \sum_{i=1}^{N-1} e^{x_{i+1} - x_i}.$$

The eigenfunctions ψ_λ of H are naturally indexed by $\lambda \in \iota\mathbb{R}^N$, given explicitly by an integral formula due to Givental (1997).

There is a positive eigenfunction ψ_0 with $H\psi_0 = 0$.

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Funny fact: $\psi_0(x)$ is the ‘volume’ of the set of ‘Gelfand-Tsetlin patterns’ with top row x, but with indicator functions $1_{a \le b}$ replaced by double exponentials $\exp(-e^{a-b})$.

The process Z_t^N

Theorem

The stochastic process $\log Z_t^N, \; t > 0$ *has the same law as the *first coordinate* of the diffusion in* $\mathbb{R}^N$ *with generator*

$$\mathcal{L} = \frac{1}{2}\psi_0^{-1} H \psi_0 = \frac{1}{2}\Delta + \nabla \log \psi_0 \cdot \nabla$$

started from $` - \infty'$.

This diffusion can be thought of as a geometric analogue of Dyson's Brownian motion.

The other coordinates

Set $X_1^N(t) = \log Z_t^N$ and, for $k = 2, \ldots, N$,

$$X_1^N(t) + \cdots + X_k^N(t) = \log \int e^{E(\phi_1)+\cdots+E(\phi_k)} d\phi_1 \ldots d\phi_k,$$

where the integral is over non-intersecting paths $\phi_1, \ldots, \phi_k$ from $(0,1), \ldots, (0,k)$ to $(t, N-k+1), \ldots, (t, N)$.

Theorem

The process X^N is a diffusion process in $\mathbb{R}^N$ with generator $\mathcal{L}$.

cf. Greene's theorem: this is based on a geometric variant of the RSK correspondence (cf. Kirillov 2000).

Generalizes a theorem of Matsumoto and Yor (1999), which in turn is a geometric analogue of Pitman's '$2M - X$' theorem.

Proof uses theory of Markov functions

Set $X^1 = B^1$. It is easy to see that $(X^1, \dots, X^N)$ is a Markov process in $\mathbb{R} \times \mathbb{R}^2 \cdots \times \mathbb{R}^N$ which satisfies a simple SDE. The Markov property of X^N follows from an intertwining relation plus some technical results concerning the entrance from '$-\infty$'.

The entrance law

The entrance law μ_t from '$-\infty$' is given by

$$\mu_t(dx) = \psi_0(x) \int_{\iota\mathbb{R}^N} \exp\left(\frac{1}{2}\sum_{i=1}^N \lambda_i^2 t\right) \psi_\lambda(x) s_N(\lambda) d\lambda,$$

where

$$s_N(\lambda) = \frac{1}{(2\pi\iota)^N N!} \prod_{j\neq k} \Gamma(\lambda_j - \lambda_k)^{-1}$$

is the *Sklyanin measure* - the Plancherel measure for the quantum Toda lattice [Sklyanin 1985, Semenov-Tian-Shanski 1994, Kharchev-Lebedev 1999].

The measure $\mu_t(dx)$ is a 'deformation' of the GUE.

The law of the partition function

Corollary

For $s > 0$,

$$Ee^{-sZ_t^N} = \int s^{\sum \lambda_i} \prod_i \Gamma(-\lambda_i)^N e^{\frac{1}{2}\sum_i \lambda_i^2 t} s_N(\lambda) d\lambda,$$

where the integral is along vertical lines with $\Re\lambda_i < 0$ for all i.

This uses a remarkable identity, conjectured by Bump (1989), proved by Stade (2002), and extended / elucidated in the present context by Gerasimov, Lebedev and Oblezin (2008). Moreover, the RHS is a Fredholm determinant.

Connection with random matrices

The probability measure on $\iota\mathbb{R}^N$ with density proportional to

$$e^{\sum_i \lambda_i^2 t/2} s_N(\lambda) \equiv \frac{1}{(2\pi\iota)^N N!} e^{\sum_i \lambda_i^2 t/2} \prod_{i>j} (\lambda_i - \lambda_j) \prod_{i<j} \frac{\sin \pi(\lambda_i - \lambda_j)}{\pi}$$

is (up to a factor of $\iota\pi$) the law, at time $1/t$, of the radial part of a Brownian motion in the symmetric space of positive definite Hermitian matrices. In particular, it is a determinantal point process, so $Ee^{-sZ_t^N}$ can be written as a Fredholm determinant.

Crossover distributions

The law of $\log Z_t^N$ should converge (in an appropriate scaling) to the 'crossover distributions' recently introduced in the context of the KPZ / stochastic heat equation by Sassamoto-Spohn (2010) and Amir-Corwin-Quastel (2010) - building on recent work of Tracy and Widom on ASEP - and also via a different approach by Dotsenko-Klumov (2010).

The above RSK-type construction extends naturally to the continuum setting.