LITTELMANN PATHS AND BROWNIAN PATHS

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Abstract
We study some path transformations related to Pitman's theorem [28, Th. 1.3] on Brownian motion and the three-dimensional Bessel process. We relate these to the Littelmann path model (see [22]) and give applications to representation theory and to Brownian motion in a Weyl chamber.

Contents

1. Introduction

Some transformations defined on continuous paths with values in a vector space have appeared in recent years in two separate parts of mathematics. On the one hand, Littelmann [22] developed his path model in order to give a unified combinatorial setup for representation theory, generalizing the theory of Young tableaux to semisimple or Kac-Moody Lie algebras of type other than A. On the other hand, in probability theory, several path transformations have been introduced that yield a construction of Brownian motion in a Weyl chamber starting from a Brownian motion in the corresponding Cartan Lie algebra. The oldest and simplest of these transformations comes from Pitman's theorem [28, Th. 1.3], which states that if $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion, then the stochastic process $R_t := B_t - 2\inf_{0\leq s\leq t} B_s$ is a three-dimensional Bessel process; that is, it is distributed as the Euclidean norm of a three-dimensional Brownian motion. (Actually, Pitman stated his theorem with the transformation $2\sup_{0\leq s\leq t} B_s - B_t$, but thanks to the symmetry of Brownian motion, this is clearly equivalent to the above statement.) It turns out that the fact that, here, the dimension of the Brownian motion is equal to one, the rank of the group SU(2),

DUKE MATHEMATICAL JOURNAL
Vol. 130, No. 1, © 2005
Received 10 March 2004. Revision received 20 October 2004.
2000 *Mathematics Subject Classification*. Primary 20F55, 60J65; Secondary 17B10.

while three, the dimension of the Bessel process, is the dimension of the group SU(2), is not a mere coincidence but a fundamental fact that we clarify in the following. Pitman's theorem has been extended in several ways. The first step has been the result of Gravner, Tracy, and Widom [15] and of Baryshnikov [1]; it states that the largest eigenvalue of a random $(n \times n)$-Hermitian matrix in the Gaussian unitary ensemble (GUE) is distributed as the random variable

$$\sup_{1=t_n \geq t_{n-1} \geq \cdots \geq t_1 \geq t_0 = 0} \sum_{i=1}^{n} \big(B_i(t_i) - B_i(t_{i-1})\big),$$

where $(B_1, \ldots, B_n)$ is a standard n-dimensional Brownian motion. This result in turn was generalized in [7] and [27]. These extensions involve path transformations that generalize Pitman's transform and are closely related to the Littelmann path model. One of the purposes of this article is to clarify these connections as well as to settle a number of questions raised in these works. In the course of these investigations, we derive several applications to representation theory. These path transformations occur in quite different contexts since the one in [7] is expressed by representation-theoretic means, whereas the one in [27] is purely combinatorial and arises from queuing theory considerations.

Let us describe more precisely the content of the article. We start by defining the Pitman transforms, which are the main object of study in this article. These transforms operate on the set of continuous functions $\pi : [0, T] \to V$ with values in some real vector space V such that $\pi(0) = 0$. They are given by the formula

$$\mathscr{P}_\alpha \pi(t) = \pi(t) - \inf_{t \geq s \geq 0} \alpha^\vee\big(\pi(s)\big)\alpha, \quad t \in [0, T].$$

Here $\alpha \in V$ and $\alpha^\vee \in V^\vee$ (where $V^\vee$ is the dual space of V) satisfy $\alpha^\vee(\alpha) = 2$. These are multidimensional generalizations of the transform occurring in Pitman's theorem. They are related to Littelmann's operators, as shown in Section 2.2. We show that these transforms satisfy braid relations; that is, if $\alpha, \beta \in V$ and $\alpha^\vee, \beta^\vee \in V^\vee$ are such that $\alpha^\vee(\alpha) = \beta^\vee(\beta) = 2$, $\alpha^\vee(\beta) < 0$, $\beta^\vee(\alpha) < 0$, and $\alpha^\vee(\beta)\beta^\vee(\alpha) = 4\cos^2(\pi/n)$, where $n \geq 2$ is some integer, then one has

$$\mathscr{P}_\alpha \mathscr{P}_\beta \mathscr{P}_\alpha \cdots = \mathscr{P}_\beta \mathscr{P}_\alpha \mathscr{P}_\beta \cdots,$$

where there are n factors in each product. Consider now a Coxeter system (W, S) (see [8], [18]). To each fundamental reflection s_i we associate a Pitman transform $\mathscr{P}_{\alpha_i}$. The braid relations imply that if $w \in W$ has a reduced decomposition $w = s_1 \cdots s_n$, then the operator $\mathscr{P}_w = \mathscr{P}_{\alpha_1} \cdots \mathscr{P}_{\alpha_n}$ is well defined; that is, it depends only on w and not on the reduced decomposition. We show that if W is a Weyl group, $w_0 \in W$ is the longest element, and if π is a dominant path ending in the weight lattice, then for any path η in the Littelmann module generated by π, one has

$$\pi = \mathscr{P}_{w_0}\eta. \tag{1.1}$$

The path transformation introduced in [27] can be expressed as $\mathscr{P}_{w_0}$, where w_0 is the longest element in the Coxeter group of type A.

We derive a representation-theoretic formula for $\mathscr{P}_w$, in the case of a Weyl group, expressed in terms of representations of the Langlands dual group (see Theorem 3.12). This formula is canonical in the sense that it is independent of any choice of a reduced decomposition of w in the Weyl group. It is obtained by lifting the path to a path $g(t)$ with values in the Borel subgroup of the simply connected complex Lie group associated with the root system. Then one obtains integral transformations, which relate the diagonal parts in the Gauss decompositions of the elements $\overline{w}g(t)$. The Pitman transforms are obtained by going down to the Cartan algebra by applying the Laplace method. By (1.1), we obtain in this way a new formula for the dominant path in some Littelmann module, in terms of any of the paths of the module, which is a generalization to arbitrary root systems of Greene's formula (see [14]). As a by-product of this formula, we also obtain a direct proof of the symmetry of the Littlewood-Richardson coefficients.

This formula appeared in [7], where it was conjectured that the associated map transforms a Brownian motion in the Cartan Lie algebra into a Brownian motion in the Weyl chamber. This conjecture was proved in [7] for some classical groups. Here we give a completely different proof, valid for all root systems.

This article is organized as follows. In Section 2, we define the elementary Pitman transformations operating on continuous paths with values in some real vector space V, taking the value zero at zero. The first result is a formula for the repeated compositions of two Pitman transforms which implies that they satisfy the braid relations. Then we define Pitman transformations $\mathscr{P}_w$ associated to a Coxeter system (W, S). In Section 3, we prove our main result, which is a representation-theoretic formula for these operators $\mathscr{P}_w$ in the case where W is a Weyl group. This formula unifies the results of [27] and [7]. Results of Berenstein and Zelevinsky [2] and Fomin and Zelevinsky [13] on totally positive matrices play a crucial role in the proof. In Section 4, we make some comments on a duality transformation naturally defined on paths, which generalizes the Schützenberger involution, and give an application to the symmetry of the Littlewood-Richardson rule. In Section 5, we give two proofs of the generalization of the representation of Brownian motion in a Weyl chamber obtained in [27] and [7]. One of the proofs relies essentially on the duality properties, while the other uses Littelmann paths in the context of Weyl groups. Finally, we come to the appendix, to which we have postponed a technical proof.

2. Braid relations for the Pitman transforms

2.1. Pitman transforms

Let V be a real vector space with dual space $V^\vee$. Let $\alpha \in V$ and $\alpha^\vee \in V^\vee$ be such that $\alpha^\vee(\alpha) = 2$.

Definition 2.1

The Pitman transform $\mathscr{P}_\alpha$ is defined on the set of continuous paths $\pi : [0, T] \to V$, satisfying $\pi(0) = 0$, by the formula

$$\mathscr{P}_\alpha \pi(t) = \pi(t) - \inf_{t \geq s \geq 0} \alpha^\vee\bigl(\pi(s)\bigr)\alpha, \qquad T \geq t \geq 0.$$

This transformation seems to have appeared for the first time in [28] in the one-dimensional case. Note that $\mathscr{P}_\alpha$ actually depends on the pair $(\alpha, \alpha^\vee)$. For simplicity, we use the notation $\mathscr{P}_\alpha$; it is always clear from the context which $\alpha^\vee$ is involved.

When, for some $v \in V$, π is the linear path $\pi(t) = tv$, then $\mathscr{P}_\alpha \pi = \pi$ when $\alpha^\vee(v) \geq 0$ and $\mathscr{P}_\alpha \pi = s_\alpha \pi$ when $\alpha^\vee(v) \leq 0$, where s_α is the reflection on V:

$$s_\alpha v = v - \alpha^\vee(v)\alpha. \tag{2.1}$$

PROPOSITION 2.2

The Pitman transforms satisfy the following properties.

(i) *For any $\lambda > 0$, the Pitman transformation associated with the pair $(\lambda\alpha, \alpha^\vee/\lambda)$ is the same as the one associated with the pair $(\alpha, \alpha^\vee)$.*

(ii) *One has $\alpha^\vee(\mathscr{P}_\alpha \pi(t)) \geq 0$ for all $t \in [0, T]$. Furthermore, $\mathscr{P}_\alpha \pi = \pi$ if and only if $\alpha^\vee(\pi(t)) \geq 0$ for all $t \in [0, T]$.*

(iii) *The transformation $\mathscr{P}_\alpha$ is an idempotent; that is, $\mathscr{P}_\alpha \mathscr{P}_\alpha \pi = \mathscr{P}_\alpha \pi$ for all π.*

(iv) *Let $\pi : [0, \infty[\to V$ be a path; then $-\inf_{0 \leq t \leq T} \alpha^\vee(\pi(t)) \in [0, \alpha^\vee(\mathscr{P}_\alpha \pi(T))]$. Conversely, given a path η satisfying $\eta(0) = 0$, $\alpha^\vee(\eta(t)) \geq 0$ for all $t \in [0, T]$, and $x \in [0, \alpha^\vee(\eta(T))]$, there exists a unique path π such that $\mathscr{P}_\alpha \pi = \eta$ and $x = -\inf_{T \geq t \geq 0} \alpha^\vee(\pi(t))$. Actually, π is given by the formula*

$$\pi(t) = \eta(t) - \min\Bigl(x, \inf_{T \geq s \geq t} \alpha^\vee(\eta(s))\Bigr)\alpha. \tag{2.2}$$

Proof

Items (i) and (ii) are trivial, and (iii) follows immediately from (ii). Hopefully, the reader can give a formal proof of (iv) (see the appendix for such a proof), but it is perhaps more illuminating to stare for a few minutes at Figure 1, which shows, in the one-dimensional case with $\alpha = 1$, $\alpha^\vee = 2$, the graph of a function $g : [0, 1] \to \mathbb{R}$ as well as those of I, $-I$, and $f = \mathscr{P}_\alpha g$, where $I(s) = \inf_{0 \leq u \leq s} g(u)$. $\square$

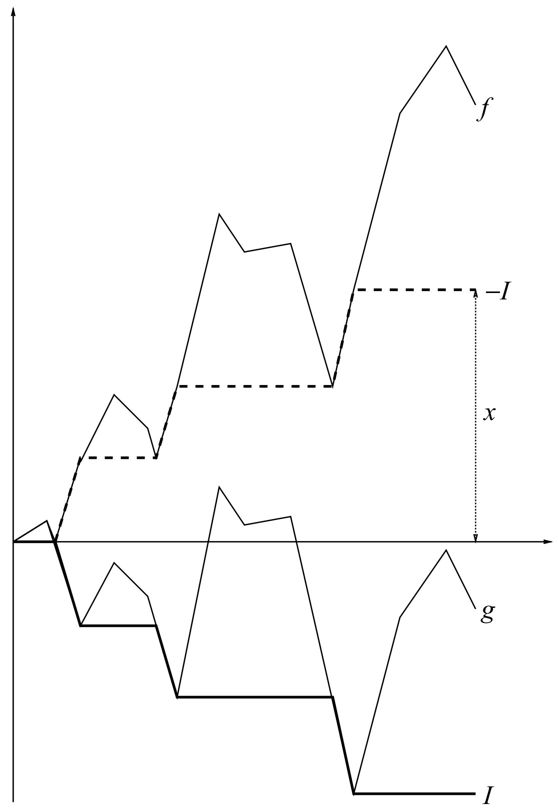


Figure 1

2.2. Relation with Littelmann path operators

Using Proposition 2.2(iv), we can define generalized Littelmann transformations. Recall that Littelmann operators are defined on paths with values in the dual space $\mathfrak{a}^*$ of some real Lie algebra $\mathfrak{a}$. The image of a path is either another path or the symbol $\mathbf{0}$ (actually the zero element in the $\mathbb{Z}$-module generated by all paths). We define continuous versions of these operators.

Definition 2.3

Let $\pi : [0, T] \to V$ be a continuous path satisfying $\pi(0) = 0$, and let $x \in \mathbb{R}$; then $E_\alpha^x \pi$ is the unique path such that

$$\mathscr{P}_\alpha E_\alpha^x \pi = \mathscr{P}_\alpha \pi \qquad \text{and} \qquad \alpha^\vee\big(E_\alpha^x \pi(T)\big) = \alpha^\vee\big(\pi(T)\big) + x$$

if $-2\alpha^\vee(\pi(T)) + 2\inf_{0\le t\le T} \alpha^\vee(\pi(t)) \le x \le -2\inf_{0\le t\le T} \alpha^\vee(\pi(t))$ and $E_\alpha^x \pi = \mathbf{0}$ otherwise.

One easily checks that $E_\alpha^0\pi = \pi$ and $E_\alpha^x E_\alpha^y \pi = E_\alpha^{x+y}\pi$ as long as $E_\alpha^y\pi \neq \mathbf{0}$. When α is a root and $\alpha^\vee$ its coroot in some root system, then E_α^2 and E_α^{-2} coincide with the Littelmann operators e_α and f_α defined in [22]. Recall that a path π is called integral if its endpoint $\pi(T)$ is in the weight lattice and, for each simple root α, the minimum of the function $\alpha^\vee(\pi(t))$ is an integer. The class of integral paths is invariant under the Littelmann operators. For such paths, the action of a Pitman transform can be expressed through Littelmann operators by

$$\mathscr{P}_\alpha\pi = e_\alpha^{n_\alpha}(\pi), \tag{2.3}$$

where n_α is the largest integer n such that $e_\alpha^n(\pi) \neq \mathbf{0}$.

2.3. Braid relations

An important property of the Pitman transforms is the following result.

THEOREM 2.4

Let $\alpha, \beta \in V$, and let $\alpha^\vee, \beta^\vee \in V^\vee$ be such that $\alpha^\vee(\alpha) = \beta^\vee(\beta) = 2$, $\alpha^\vee(\beta) < 0$, $\beta^\vee(\alpha) < 0$ $\alpha^\vee(\beta)\beta^\vee(\alpha) = 4\cos^2(\pi/n)$, where $n \geq 2$ is some integer; then one has

$$\mathscr{P}_\alpha\mathscr{P}_\beta\mathscr{P}_\alpha \cdots = \mathscr{P}_\beta\mathscr{P}_\alpha\mathscr{P}_\beta \cdots ,$$

where there are n factors in each product.

We prove Theorem 2.4 as a corollary to the result of Section 2.4. Note that if $\alpha^\vee(\beta) = \beta^\vee(\alpha) = 0$, then $\mathscr{P}_\alpha\mathscr{P}_\beta = \mathscr{P}_\beta\mathscr{P}_\alpha$ by a simple computation. For crystallographic angles (i.e., $n = 2, 3, 4, 6$), a proof of Theorem 2.4 could also be deduced from Littelmann's theory (see [23] or [19]). We provide still another (and hopefully more conceptual) proof for these angles in Section 3 (see Remark 3.10). The general case seems to be new.

2.4. A formula for $\mathscr{P}_\alpha\mathscr{P}_\beta\mathscr{P}_\alpha\mathscr{P}_\beta\cdots$

Let $\alpha, \beta \in V$, and let $\alpha^\vee, \beta^\vee \in V^\vee$ be such that $\alpha^\vee(\beta) < 0$ and $\beta^\vee(\alpha) < 0$. By Proposition 2.2(i), we can—and do—assume by rescaling that $\alpha^\vee(\beta) = \beta^\vee(\alpha)$ without changing $\mathscr{P}_\alpha$ and $\mathscr{P}_\beta$. We use the notation

$$\rho = -\frac{1}{2}\alpha^\vee(\beta) = -\frac{1}{2}\beta^\vee(\alpha), \qquad X(s) = \alpha^\vee\big(\pi(s)\big), \qquad Y(s) = \beta^\vee\big(\pi(s)\big).$$

THEOREM 2.5
Let n be a positive integer; if $\rho \geq \cos(\pi/n)$, then one has

$$\underbrace{(\mathscr{P}_\alpha\mathscr{P}_\beta\mathscr{P}_\alpha\cdots)}_{n \text{ terms}}\pi(t) = \pi(t) - \inf_{t\geq s_0\geq s_1\geq\cdots\geq s_{n-1}\geq 0}\Big(\sum_{i=0}^{n-1} T_i(\rho)Z^{(i)}(s_i)\Big)\alpha$$
$$- \inf_{t\geq s_0\geq s_1\geq\cdots\geq s_{n-2}\geq 0}\Big(\sum_{i=0}^{n-2} T_i(\rho)Z^{(i+1)}(s_i)\Big)\beta, \qquad (2.4)$$

where $Z^{(k)} = X$ if k is even and $Z^{(k)} = Y$ if k is odd. The $T_k(x)$ are the Tchebycheff polynomials defined by $T_0(x) = 1$, $T_1(x) = 2x$, and $2xT_k(x) = T_{k-1}(x) + T_{k+1}(x)$ for $k \geq 1$.

The Tchebycheff polynomials satisfy $T_k(\cos\theta) = (\sin(k+1)\theta)/\sin\theta$; and, in particular, under the assumptions on ρ and n, one has $T_k(\rho) \geq 0$ for all $k \leq n-1$.

Assuming Theorem 2.5, we obtain Theorem 2.4.

Proof of Theorem 2.4
Let $\alpha^\vee(\beta) = \beta^\vee(\alpha) = -2\cos(\pi/n)$; then one has $T_{n-1}(\rho) = 0$, and the last term in the coefficient of α in the right-hand side of (2.4) vanishes. It follows by inspection that this term equals the coefficient of α in the analogous formula for $\underbrace{\mathscr{P}_\beta\mathscr{P}_\alpha\mathscr{P}_\beta\cdots}_{n \text{ terms}}\pi(t)$.
A similar argument works for the coefficient of β. □

The proof of Theorem 2.5 is by induction on n. It is easy to check the formula for $n = 1$ or 2. We do the induction in Sections 2.5 and 2.6.

See Figure 2 for a picture of the case $\rho = 1/2$.

2.5. Two intermediate lemmas

LEMMA 2.6
Let $X : [0, t] \to \mathbb{R}$ be a continuous function with $X(0) = 0$, and let

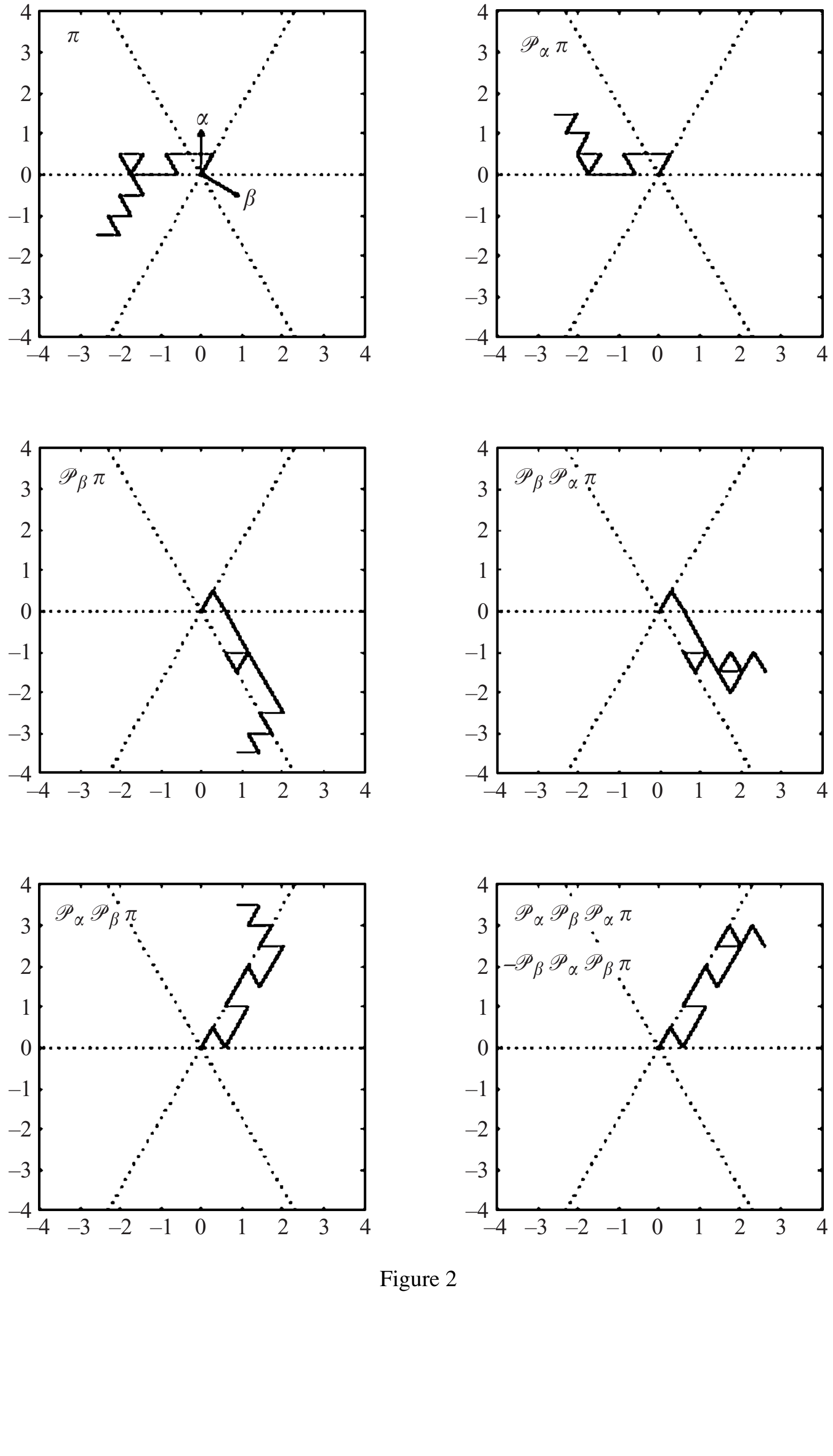
$$t_0 = \sup\{s \geq 0 \mid X_s = \inf_{s\geq u\geq 0} X_u\}.$$

Then for all $u \leq t_0$, one has

$$\inf_{t\geq s\geq u}\big(X(s) - 2\inf_{s\geq w\geq 0} X(w)\big) = -\inf_{u\geq v\geq 0} X(v).$$

Proof
This is obtained as a by-product of the proof in Section 6. Again, it is perhaps more convincing to stare at Figure 1 than to give a formal proof. □

Elaborating on this, we obtain the next result.



π
α
β
$\mathscr{P}_\alpha \pi$
$\mathscr{P}_\beta \pi$
$\mathscr{P}_\beta \mathscr{P}_\alpha \pi$
$\mathscr{P}_\alpha \mathscr{P}_\beta \pi$
$\mathscr{P}_\alpha \mathscr{P}_\beta \mathscr{P}_\alpha \pi$
$= \mathscr{P}_\beta \mathscr{P}_\alpha \mathscr{P}_\beta \pi$

Figure 2

LEMMA 2.7
Let X and Y be continuous functions such that $X(0) = Y(0) = 0$; then

$$\begin{aligned}
&\inf_{t\geq s\geq 0}\big(X(s) + \inf_{s\geq u\geq 0} Y(u)\big)\\
&\quad = \inf_{t\geq s\geq 0} X(s) + \inf_{t\geq s\geq 0}\big(X(s) - 2\inf_{s\geq u\geq 0} X(u) + \inf_{s\geq u\geq 0}(Y(u) + \inf_{u\geq v\geq 0} X(v))\big).
\end{aligned}$$

Proof
The first term is $I = \inf_{t\geq s\geq u\geq 0}(X(s) + Y(u))$. Let t_0 be, as in Lemma 2.6, the last time when X reaches its minimum over $[0, t]$; then

$$I = \inf\big(\inf_{t_0\geq u\geq 0}(Y(u) + X(t_0));\ \inf_{t\geq s\geq u\geq t_0\geq 0}(Y(u) + X(s))\big).$$

Let J be the second term in the identity to be proved; then

$$J = \inf_{t\geq s\geq 0}\big[X(s) - 2\inf_{s\geq u\geq 0} X(u) + \inf_{s\geq u\geq 0}(Y(u) + \inf_{u\geq v\geq 0} X(v))\big] + X(t_0).$$

Introduce again the time t_0; then

$$\begin{aligned}
J &= \inf_{t\geq s\geq u\geq 0}\big(X(s) - 2\inf_{s\geq w\geq 0} X(w) + Y(u) + \inf_{u\geq v\geq 0} X(v)\big) + X(t_0)\\
&= \inf\big(\inf_{\substack{t\geq s\geq u\geq 0\\ t_0\geq u}}(Y(u) + X(s) - 2\inf_{s\geq w\geq 0} X(w) + \inf_{u\geq v\geq 0} X(v) + X(t_0));\\
&\qquad\quad \inf_{t\geq s\geq u\geq t_0}(Y(u) + X(s) - 2\inf_{s\geq w\geq 0} X(w) + \inf_{u\geq v\geq 0} X(v) + X(t_0))\big).
\end{aligned}$$

But if $u \leq t_0$, then by Lemma 2.6, one has $\inf_{t\geq s\geq u}(X(s) - 2\inf_{s\geq w\geq 0} X(w)) = -\inf_{u\geq v\geq 0} X(v)$. If $t_0 \leq u$, then $\inf_{s\geq w\geq 0} X(w) = X(t_0)$; therefore

$$\begin{aligned}
J &= \inf\big(\inf_{t_0\geq u\geq 0}(Y(u) + X(t_0));\ \inf_{0\geq u\geq t_0}(Y(u) + \inf_{t\geq s\geq u} X(s))\big)\\
&= \inf\big(\inf_{t_0\geq u\geq 0}(Y(u) + X(t_0));\ \inf_{t\geq s\geq u\geq t_0\geq 0}(Y(u) + X(s))\big)\\
&= I.
\end{aligned}$$

□

2.6. Proof of Theorem 2.5
Assume that the result of the theorem holds for some n with n even. Then

$$\underbrace{\mathcal{P}_\alpha\mathcal{P}_\beta\mathcal{P}_\alpha\cdots}_{n+1 \text{ terms}} = \underbrace{\mathcal{P}_\alpha\mathcal{P}_\beta\mathcal{P}_\alpha\cdots}_{n \text{ terms}}\mathcal{P}_\alpha,$$

and one has

$$\begin{aligned}
\alpha^\vee\big(\mathcal{P}_\alpha\pi(s)\big) &= X(s) - 2\inf_{s\geq u\geq 0} X(u),\\
\beta^\vee\big(\mathcal{P}_\alpha\pi(s)\big) &= Y(s) + 2\rho\inf_{s\geq u\geq 0} X(u).
\end{aligned}$$

Therefore, by the induction hypothesis,

$$\begin{aligned}\underbrace{\mathscr{P}_\alpha\mathscr{P}_\beta\mathscr{P}_\alpha\cdots}_{n+1\text{ terms}}\pi(t) &= \underbrace{\mathscr{P}_\alpha\mathscr{P}_\beta\mathscr{P}_\alpha\cdots}_{n\text{ terms}}(\mathscr{P}_\alpha\pi)(t)\\ &= \pi(t)-\inf_{t\geq s\geq 0}X(s)\alpha-\inf_{t\geq s_0\geq s_1\geq\cdots\geq s_{n-1}\geq 0}\Big(\sum_{i=0}^{n-1}T_i(\rho)\hat{Z}^{(i)}(s_i)\Big)\alpha\\ &\quad-\inf_{t\geq s_0\geq s_1\geq\cdots\geq s_{n-2}\geq 0}\Big(\sum_{i=0}^{n-2}T_i(\rho)\hat{Z}^{(i+1)}(s_i)\Big)\beta,\end{aligned}$$

where

$$\hat{Z}_\alpha^{(i)}(s)=\begin{cases}X(s)-2\inf_{s\geq u\geq 0}X(u) & \text{for } i \text{ even},\\ Y(s)+2\rho\inf_{s\geq u\geq 0}X(u) & \text{for } i \text{ odd}.\end{cases}$$

The coefficient of α in the above expression has the form

$$\begin{aligned}H_\alpha &= -\inf_{t\geq s\geq 0}T_0(\rho)X(s)\\ &\quad-\inf_{t\geq s\geq 0}\big(T_0(\rho)X(s)-2\inf_{s\geq u\geq 0}T_0(\rho)X(u)+\inf_{s\geq u\geq 0}(\Gamma(u)+\inf_{u\geq v\geq 0}T_0(\rho)X(v))\big),\end{aligned}$$

where

$$\begin{aligned}\Gamma(u) &= T_1(\rho)Y(u)+2\rho T_1(\rho)\inf_{u\geq v\geq 0}X(v)\\ &\quad+\inf_{u\geq u_2\geq u_3\geq\cdots\geq u_{n-1}\geq 0}\Big(\sum_{i=2}^{n-1}T_i(\rho)\hat{Z}^{(i)}(u_i)\Big)-T_0(\rho)\inf_{u\geq v\geq 0}X(v)\\ &= T_1(\rho)Y(u)+T_2(\rho)\inf_{u\geq v\geq 0}X(v)+\inf_{u\geq u_2\geq u_3\geq\cdots\geq u_{n-1}\geq 0}\Big(\sum_{i=2}^{n-1}T_i(\rho)\hat{Z}^{(i)}\Big),\end{aligned}$$

so that we can apply Lemma 2.7 to transform it into

$$H_\alpha=-\inf_{t\geq s\geq 0}\big(T_0(\rho)X(s)+\inf_{s\geq u\geq 0}\Gamma(u)\big).$$

Let us prove by induction on k that

$$H_\alpha=-\inf_{t\geq u_0\geq u_1\geq\cdots\geq u_{2k}}\Big(\sum_{i=0}^{2k}T_i(\rho)Z^{(i)}(u_i)+W_k(u_{2k-1})\Big)$$

with

$$W_k(v)=\inf_{v\geq u_{2k}\geq u_{2k+1}\geq\cdots\geq u_{n-1}\geq 0}\Big(\sum_{i=2k}^{n-1}T_i(\rho)\hat{Z}^{(i)}(u_i)\Big).$$

Indeed, the formula holds for $k = 1$ by the computation above. Assume that this holds for some k; then one has

$$
\begin{aligned}
H_\alpha &= -\inf_{t\geq u_0\geq u_1\geq\cdots\geq u_{2k}} \Big(\sum_{i=0}^{2k} T_i(\rho)Z^{(i)}(u_i) + W_k(u_{2k-1})\Big)\\
&= -\inf_{t\geq u_1\geq u_2\geq\cdots\geq u_{2k-1}} \Big(\sum_{i=0}^{2k-1} T_i(\rho)Z^{(i)}(u_i) + \inf_{u_{2k-1}\geq v\geq 0} T_{2k}(\rho)X(v)\\
&\qquad + {}_{u_{2k-1}\geq v\geq 0}\big((T_{2k}(\rho)X(v) - 2\inf_{v\geq w\geq 0} T_{2k}(\rho)X(w))\\
&\qquad + \inf_{w\geq z\geq 0}(R_k(z) + \inf_{z\geq\tau\geq 0} T_{2k}(\rho)X(\tau))\big)\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
R_k(z) &= T_{2k+1}(\rho)Y(z) + 2\rho T_{2k+1}(\rho)\inf_{z\geq\tau\geq 0} X(\tau)\\
&\qquad + \inf_{z\geq u_{2k+2}\geq\cdots u_{n-1}} \Big(\sum_{i=2k+2}^{n-1} \hat{Z}^{(i)}(u_i)\Big) - \inf_{z\geq\tau\geq 0} T_{2k}(\rho)X(\tau)\\
&= T_{2k+1}(\rho)Y(z) + T_{2k+2}(\rho)\inf_{z\geq\tau\geq 0} X(\tau) + \inf_{z\geq u_{2k+2}\geq\cdots u_{n-1}} \Big(\sum_{i=2k+2}^{n-1} \hat{Z}^{(i)}(u_i)\Big),
\end{aligned}
$$

where we used $2\rho T_{2k+1}(\rho) - T_{2k}(\rho) = T_{2k+2}(\rho)$. Applying Lemma 2.7, we get

$$
\begin{aligned}
H_\alpha &= -\inf_{t\geq u_1\geq u_2\geq\cdots\geq u_{2k-1}} \Big(\sum_{i=0}^{2k-1} T_i(\rho)Z^{(i)}(u_i) + \inf_{u_{2k-1}\geq v\geq 0}\big(T_{2k}(\rho)X(v) + \inf_{w\geq z\geq 0} R_k(z)\big)\Big)\\
&= -\inf_{t\geq u_0\geq u_1\geq\cdots\geq u_{2k+2}} \Big(\sum_{i=0}^{2k+2} T_i(\rho)Z^{(i)}(u_i) + W_{k+1}(u_{2k+1})\Big).
\end{aligned}
$$

Taking $k = n$ gives the required formula for H_α. For the coefficient of β, note that

$$
\underbrace{\mathscr{P}_\alpha\mathscr{P}_\beta\mathscr{P}_\alpha\cdots}_{n+1\text{ terms}}\pi(t) = \mathscr{P}_\alpha(\underbrace{\mathscr{P}_\beta\mathscr{P}_\alpha\mathscr{P}_\beta\cdots}_{n\text{ terms}}\pi)(t)
$$

and that the formula for $n + 1$ follows immediately from the formula at step n for $\underbrace{\mathscr{P}_\beta\mathscr{P}_\alpha\mathscr{P}_\beta\cdots}_{n\text{ terms}}$. The case where n is odd is treated in a similar way. □

2.7. Pitman transformations for Coxeter and Weyl groups

Let W be a Coxeter group; that is, W is generated by a finite set S of reflections of a real vector space V and (W, S) is a Coxeter system (see [8], [18]). For each $s \in S$, let $\alpha_s \in V$, and let $\alpha_s^\vee \in V^\vee$, where $V^\vee$ is the dual space of V such that $s = s_{\alpha_s}$ is the reflection associated to α_s (see (2.1)). Then α_s is called the simple root associated with $s \in S$ and $\alpha_s^\vee$ is called its coroot.

Denote by $\mathscr{P}_s$ the Pitman transform associated with the pair $(\alpha_s, \alpha_s^\vee)$. By the results of the preceding sections, the $\mathscr{P}_s$, $s \in S$, form a representation of the monoid generated by idempotents satisfying the braid relations. Such a monoid occurs in the theory of Hecke algebras for $q = 0$ and in the calculus of Borel orbits (see, e.g., [20], where this monoid is called the Richardson-Springer monoid).

Let H_s be the closed half-space $H_s = \{v \in V | \alpha_s^\vee(v) \geq 0\}$. Let $w \in W$, and let $w = s_1 \cdots s_l$ be a reduced decomposition of w, where $l = l(w)$ is the length of w. By Theorem 2.4 and a fundamental result of Matsumoto in [8, Ch. IV, No. 1.5, Prop. 5], the operator $\mathscr{P}_{s_1} \cdots \mathscr{P}_{s_l}$ depends only on w and not on the chosen reduced decomposition. We denote this operator by $\mathscr{P}_w$.

PROPOSITION 2.8

Let $w \in W$, let $L_w = \{s \in S \,|\, l(sw) < l(w)\}$, and let $R_w = \{s \in S \,|\, l(ws) < l(w)\}$. For any path π, the path $\mathscr{P}_w \pi$ lies in the convex cone $\bigcap_{s \in L_w} H_s$; one has $\mathscr{P}_s \mathscr{P}_w = \mathscr{P}_w$ for all $s \in L_w$ and $\mathscr{P}_w \mathscr{P}_s = \mathscr{P}_w$ for all $s \in R_w$.

Proof

If $l(sw) < l(w)$, then w has a reduced decomposition $w = ss_1 \cdots s_k$; therefore $\mathscr{P}_w = \mathscr{P}_s \mathscr{P}_{s_1} \cdots \mathscr{P}_{s_k}$ and $\mathscr{P}_w \pi = \mathscr{P}_s(\mathscr{P}_{s_1} \cdots \mathscr{P}_{s_k} \pi)$ lies in H_s by Proposition 2.2(ii). Furthermore, one has $\mathscr{P}_s \mathscr{P}_w = \mathscr{P}_w$ since $\mathscr{P}_s$ is an involution (see Proposition 2.2(ii)). Similarly, $\mathscr{P}_w \mathscr{P}_s = \mathscr{P}_w$ when $l(ws) < l(w)$. $\square$

COROLLARY 2.9

If W is finite and w_0 is the longest element, then $\mathscr{P}_{w_0} \pi$ takes values in the closed Weyl chamber $\overline{C} = \bigcap_{s \in S} H_s$. Furthermore, $\mathscr{P}_{w_0}$ is an idempotent and $\mathscr{P}_w \mathscr{P}_{w_0} = \mathscr{P}_{w_0} \mathscr{P}_w = \mathscr{P}_{w_0}$ for all $w \in W$.

Assume now that W is a finite Weyl group associated with a weight lattice in V. Recall that paths taking values in the Weyl chamber $\overline{C}$ are called *dominant paths* in [22] and that the set $B\pi$ of all (nonzero) paths obtained by applying products of Littelmann operators to a dominant path π is called the *Littelmann module*. From the connection between Pitman's and Littelmann's operators given in Section 2.2, we deduce the following (see also [23]).

COROLLARY 2.10

Let π be a dominant integral path; then a path η belongs to the Littelmann module $B\pi$ if and only if η is integral and $\pi = \mathscr{P}_{w_0} \eta$.

Proof
Indeed, for any path η and x such that $E_\alpha^x\eta \neq \mathbf{0}$, one has $\mathscr{P}_\alpha E_\alpha^x\eta = \mathscr{P}_\alpha\eta$; therefore $\mathscr{P}_{w_0}E_\alpha^x\eta = \mathscr{P}_{w_0}\mathscr{P}_\alpha E_\alpha^x\eta = \mathscr{P}_{w_0}\eta$. It follows that the set of paths whose image by $\mathscr{P}_{w_0}$ is π is stable under the action of Littelmann operators. If η is an integral path such that $\mathscr{P}_{w_0}\eta = \pi$ and $w_0 = s_1 \cdots s_n$ is a reduced decomposition, then by Section 2.2, the sequence $\eta, \mathscr{P}_{\alpha_n}\eta, \mathscr{P}_{\alpha_{n-1}}\mathscr{P}_{\alpha_n}\eta, \dots, \pi$ is obtained by successive applications of Littelmann operators; therefore they all belong to the Littelmann module $B\pi$. □

Let us come back to the general case of a finite Coxeter group. We now study the set of all paths η such that $\mathscr{P}_w\eta$ is a given dominant path. Let $w = s_1 \cdots s_q$ be a reduced decomposition. Let η be a path such that $\eta(0) = 0$ and $\pi = \mathscr{P}_w\eta$ is a dominant path. Denote $\eta_0 = \pi$, $\eta_q = \eta$, and $\eta_j = \mathscr{P}_{s_{j+1}} \cdots \mathscr{P}_{s_q}\eta_q$ for $j = 1, 2, \dots, q-1$. Then by Proposition 2.2(iv) for all $j = 1, 2, \dots, q$, the path η_j is uniquely specified among paths γ such that $\mathscr{P}_{s_j}\gamma = \eta_{j-1}$ by the number $x_j = -\inf_{0\le t\le T}\alpha_{s_j}^\vee(\eta_j(t)) \in [0, \alpha_{s_j}^\vee(\eta_{j-1}(T))]$. It follows that $\eta = \eta_q$ is uniquely specified among all paths γ such that $\mathscr{P}_{w_0}\gamma = \pi$ by the sequence $x_1, x_2, \dots, x_q$. These coordinates are subject to the inequalities $0 \le x_j \le \alpha_{s_j}^\vee(\eta_{j-1}(T))$. From

$$\eta_{j-1}(T) = \eta_j(T) + x_j\alpha_{s_j},$$

one obtains

$$\pi(T) = \eta_0(T) = \eta_j(T) + \sum_{l=1}^{j} x_l\alpha_{s_l}.$$

Therefore the inequality $0 \le x_j \le \alpha_{s_j}^\vee(\eta_{j-1}(T))$ reads

$$0 \le x_j \le \alpha_{s_j}^\vee\big(\pi(T)\big) - \sum_{l=1}^{j-1} x_l\alpha_{s_j}^\vee(\alpha_{s_l}).$$

It follows that the set of all paths η such that $\mathscr{P}_w\eta = \pi$ can be parametrized by a subset of the convex polytope

$$K_\pi = \left\{(x_1, \dots, x_q) \in \mathbb{R}^q \;\middle|\; 0 \le x_j \le \alpha_{s_j}^\vee\big(\pi(T)\big) - \sum_{l=1}^{j-1} x_l\alpha_{s_j}^\vee(\alpha_{s_l});\ j = 1, \dots, q\right\}.$$

The path η corresponding to the point $(x_1, \dots, x_q)$ is specified by the equalities

$$\eta_{j-1}(T) = \eta_j(T) + x_j\alpha_{s_j},$$

where $\eta_j = \mathscr{P}_{s_{j+1}} \cdots \mathscr{P}_{s_q}\eta$. In the case of a Weyl group, it follows from [23] that the subset of K_π corresponding to paths η such that $\mathscr{P}_w\eta = \pi$ is the intersection of K_π with a certain convex cone that does not depend on π. This convex cone is quite difficult to describe (see [3]). Also, we do not know if a similar result holds for all finite Coxeter groups. We hope to come back to these questions in future work.

3. A representation-theoretic formula for $\mathscr{P}_w$

3.1. Semisimple groups

We recall some standard terminology. We consider a simply connected complex semisimple Lie group G associated with a root system R. Let H be a maximal torus, and let B^+, B^- be corresponding opposite Borel subgroups with unipotent radicals N^+, N^-. Let $\alpha_i, i \in I$, and $\alpha_i^\vee, i \in I$, be the simple positive roots and coroots, and let s_i be the corresponding reflections in the Weyl group W. Let $e_i, f_i, h_i, i \in I$, be Chevalley generators of the Lie algebra of G. One can choose representatives $\overline{w} \in G$ for $w \in W$ by putting $\overline{s_i} = \exp(-e_i)\exp(f_i)\exp(-e_i)$ and $\overline{vw} = \overline{v}\,\overline{w}$ if $l(v) + l(w) = l(vw)$ (see [13, Sec. 1.4]). The Lie algebra of H, denoted by $\mathfrak{h}$, has a Cartan decomposition $\mathfrak{h} = \mathfrak{a} + i\mathfrak{a}$ such that the roots α_i take real values on the real vector space $\mathfrak{a}$. Thus $\mathfrak{a}$ is generated by $\alpha_i^\vee, i \in I$, and its dual, $\mathfrak{a}^*$, is generated by $\alpha_i, i \in I$. The set of weights is the lattice $P = \{\lambda \in \mathfrak{a}^*; \lambda(\alpha_i^\vee) \in \mathbb{Z}, i \in I\}$, and the set of dominant weights is $P^+ = \{\lambda \in \mathfrak{a}^*; \lambda(\alpha_i^\vee) \in \mathbb{N}, i \in I\}$. For each $\lambda \in P^+$, choose a representation space V_λ with a highest weight vector v_λ and an invariant inner product on V_λ for which v_λ is a unit vector.

LEMMA 3.1

For any dominant weight λ, $w \in W$ and indices $i_1, \ldots, i_n \in I$, one has

$$\langle e_{i_1} \cdots e_{i_n} \overline{w} v_\lambda, v_\lambda \rangle \geq 0.$$

Proof

This is an immediate consequence of [3, Lem. 7.4]. □

Let $(\omega_i, i \in I) \in P^I$ be the fundamental weights characterized by the relations $\omega_i(\alpha_j^\vee) = \delta_{i,j}$, $j \in I$. The principal minor associated with ω_i is the function on G given by

$$\Delta^{\omega_i}(g) = \langle g v_{\omega_i}, v_{\omega_i} \rangle$$

(see [2] and [13]). If $g \in G$ has a Gauss decomposition $g = [g]_-[g]_0[g]_+$ with $[g]_- \in N^-$, $[g]_0 \in H$, and $[g]_+ \in N^+$, then one has

$$\Delta^{\omega_i}(g) = [g]_0^{\omega_i} = e^{\omega_i(\log[g]_0)}. \tag{3.1}$$

3.2. Some auxiliary path transformations

We now introduce some path transformations.

Definition 3.2
Let $n_i : [0, T] \to \mathbb{R}^+$, $i \in I$, be a family of strictly positive continuous functions, and let $a : (0, T] \to \mathfrak{a}$ be a continuous map such that

$$\int_{0^+} e^{-\alpha_i(a(s))} n_i(s)\, ds < \infty.$$

We define, for $0 < t \leq T$,

$$\mathscr{T}_{i,n} a(t) = a(t) + \log\Big(\int_0^t e^{-\alpha_i(a(s))} n_i(s)\, ds\Big)\alpha_i^\vee.$$

Observe that, in general, the maps $t \mapsto a(t)$ and $t \mapsto \mathscr{T}_{i,n} a(t)$ need not be continuous at zero. For all that follows, consideration of the case $n_i \equiv 1$ in Definition 3.2 would be sufficient for our purposes, but the proofs would be the same as the general case.

Let $R^\vee$ be the root system dual to R; namely, let the roots of $R^\vee$ be the coroots of R and vice versa, and denote by $\mathscr{P}_{\alpha_i^\vee}$, $i \in I$, the corresponding Pitman transformations on $\mathfrak{a}$. Let π be a continuous path in $\mathfrak{a}$ with $\pi(0) = 0$. For $\varepsilon > 0$, let D_ε be the dilation operator $D_\varepsilon \pi(t) = \varepsilon\pi(t)$. A simple application of the Laplace method yields

$$\mathscr{P}_{\alpha_i^\vee} \pi = \lim_{\varepsilon \to 0} D_\varepsilon \mathscr{T}_{i,n} D_\varepsilon^{-1} \pi. \tag{3.2}$$

We establish, in Section 3.4, a representation-theoretic formula for a product $\mathscr{T}_{i_k,n} \cdots \mathscr{T}_{i_1,n}$ corresponding to a minimal decomposition $w = s_{i_1} \cdots s_{i_k}$ in the Weyl group. Using this formula, we use (3.2) to get a formula for the Pitman transform.

3.3. A group-theoretic interpretation of the operators $\mathscr{T}_{i,n}$
Let a be a smooth path in $\mathfrak{a}$, and let b be the path in the Borel subgroup $B^+ = HN^+$ solution to the differential equation

$$\frac{d}{dt} b(t) = \Big(\frac{d}{dt} a(t) + \sum_{i \in I} n_i(t) e_i\Big) b(t), \quad b(0) = \mathrm{id}.$$

The following expression is easy to check.

LEMMA 3.3
We have

$$b(t) = e^{a(t)} + e^{a(t)} \sum_{k \geq 1} \sum_{i_1, \ldots, i_k \in I^k} \Big(\int_{t \geq t_1 \geq t_2 \geq \cdots \geq t_k \geq 0} e^{-\alpha_{i_1}(a(t_1))} n_{i_1}(t_1) \cdots e^{-\alpha_{i_k}(a(t_k))} n_{i_k}(t_k)\, dt_1 \cdots dt_k\Big) e_{i_1} \cdots e_{i_k}. \tag{3.3}$$

Observe that this expression is well defined in each finite-dimensional representation of G since the operators e_i are nilpotent and this sum has only a finite number of nonzero terms. It is always in this context that we use this formula.

LEMMA 3.4
For any $t > 0$ and $w \in W$, one has

$$\Delta^{\omega_i}\big(b(t)\overline{w}\big) > 0.$$

Proof
By equation (3.3), one has

$$\begin{aligned}\Delta^{\omega_i}\big(b(t)\overline{w}\big) = {} & \langle e^{a(t)}\overline{w}v_{\omega_i}, v_{\omega_i}\rangle \\ & + \sum_{r\geq 1}\sum_{i_1,\dots,i_r\in I^r}\int_{t\geq t_1\geq t_2\geq\cdots\geq t_r\geq 0}\langle e^{a(t)}e^{-\alpha_{i_1}(a(t_1))}n_{i_1}(t_1)\cdots \\ & \cdots e^{-\alpha_{i_r}(a(t_r))}n_{i_r}(t_r)e_{i_1}\cdots e_{i_r}\overline{w}v_{\omega_i}, v_{\omega_i}\rangle\, dt_1\cdots dt_r, \end{aligned} \tag{3.4}$$

which is a sum of nonnegative terms, by Lemma 3.1. Furthermore, since v_{ω_i} is a highest weight vector, there exists some sequence $i_1, \dots, i_r$ such that $e_{i_1}\cdots e_{i_r}\overline{w}v_{\omega_i}$ is a nonzero multiple of v_{ω_i} and the n_i do not vanish; therefore the sum is positive. □

It follows in particular that, according to the terminology of [13], $b(t)$ belongs to the double Bruhat cell $B_+ \cap B_- w_0 B_-$ and that $b(t)\overline{w}$ has a Gauss decomposition $b(t)\overline{w} = [b(t)\overline{w}]_-[b(t)\overline{w}]_0[b(t)\overline{w}]_+$ for all $t > 0$.

Now comes the main result of this section.

THEOREM 3.5
Let $w \in W$, and let $w = s_{i_1}\cdots s_{i_k}$ be a reduced decomposition; then the H-part in the Gauss decomposition of $b(t)\overline{w}$ is equal to

$$\exp\big(\mathcal{T}_{i_k,n}\cdots\mathcal{T}_{i_1,n}a(t)\big).$$

The fact that the path $\mathcal{T}_{i_k,n}\cdots\mathcal{T}_{i_1,n}a(t)$ is well defined is part of the theorem. By the uniqueness of the Gauss decomposition, the preceding result implies the following corollary.

COROLLARY 3.6
The path

$$\mathcal{T}_{i_k,n}\cdots\mathcal{T}_{i_1,n}a(t)$$

depends only on w and n and not on the chosen reduced decomposition of w.

We denote by $\mathscr{T}_w a$ the resulting path. (It depends on n.) We thus have

$$[b(t)\overline{w}]_0 = e^{\mathscr{T}_w a(t)}. \tag{3.5}$$

Proof of Theorem 3.5
The proof is by induction on the length of w. Let s_i be such that $l(ws_i) = l(w)+1$. We assume that the H-part of the Gauss decomposition of $b(t)\overline{w}$ is $\mathscr{T}_{i_k,n}\cdots\mathscr{T}_{i_1,n}a(t)$, as required. By (3.1), it is then enough to prove that for all $t > 0$ and $i, j \in I$, one has

$$\Delta^{\omega_j}\big(b(t)\overline{ws_i}\big) = \Delta^{\omega_i}\big(b(t)\overline{w}\big)$$

if $i \neq j$ and

$$\Delta^{\omega_i}\big(b(t)\overline{ws_i}\big) = \Delta^{\omega_i}\big(b(t)\overline{w}\big)\int_0^t e^{-\alpha_i(\mathscr{T}_w a(s))} n_i(s)\, ds.$$

The claim for $i \neq j$ follows from [13, Prop. 2.3]. It remains to check the case $i = j$.

LEMMA 3.7
We have

$$\frac{\Delta^{\omega_i}(b(t)\overline{ws_i})}{\Delta^{\omega_i}(b(t)\overline{w})} \to_{t\to 0} 0.$$

Proof
From the decomposition (3.4), the fact that all terms are positive, and the fact that the n_i are positive continuous functions, we see that as $t \to 0$, one has $\Delta^{\omega_i}(b(t)\overline{w}) \sim c_1 t^{l_1}$ and $\Delta^{\omega_i}(b(t)\overline{ws_i}) \sim c_2 t^{l_2}$ for some $c_1, c_2 > 0$, where l_1 (resp., l_2) is the number of terms in the decomposition of $\omega_i - w(\omega_i)$ (resp., $\omega_i - ws_i(\omega_i)$) as a sum of simple roots. Since $l(ws_i) > l(w)$, the weight $w(\omega_i) - ws_i(\omega_i)$ is positive, and one has $l_2 > l_1$. □

LEMMA 3.8
Let $w = s_{i_1}\cdots s_{i_k}$ be a reduced decomposition, and let $b^w(t) = [b(t)\overline{w}]_0[b(t)\overline{w}]_+$. Then one has

$$\frac{d}{dt}b^w(t) = \Big(\frac{d}{dt}\mathscr{T}_{i_k,n}\cdots\mathscr{T}_{i_1,n}a(t) + \sum_{j\in I} n_j(t)e_j\Big)b^w(t).$$

Proof
We do this by induction on the length of w. Assume that this is true for w, and let s_i be such that $l(ws_i) = l(w) + 1$. Then one has

$$\frac{d}{dt}b^w(t) = \Big(\frac{d}{dt}\mathscr{T}_w a(t) + \sum_j n_j e_j\Big)b^w(t).$$

Therefore

$$\frac{d}{dt}b^w(t)\overline{s}_i = \Big(\frac{d}{dt}\mathscr{T}_w a(t) + \sum_j n_j(t)e_j\Big)b^w(t)\overline{s}_i.$$

Since $b^w(t) \in B^+$, by [2] and [13], the Gauss decomposition of $b^w(t)\overline{s}_i$ has the form

$$b^w(t)\overline{s}_i = \exp\big(\beta(t)f_i\big)b^{ws_i}(t)$$

with $\beta(t) > 0$ for $t > 0$, and one has, since f_i commutes with all e_j for $j \neq i$,

$$\begin{aligned}
\frac{d}{dt}b^{ws_i}(t) &= \frac{d}{dt}\big[\exp(-\beta(t)f_i)b^w(t)\overline{s}_i\big] \\
&= -\Big(\frac{d}{dt}\beta(t)\Big)f_i \exp\big(-\beta(t)f_i\big)b^w(t)\overline{s}_i \\
&\quad + \exp\big(-\beta(t)f_i\big)\Big(\frac{d}{dt}\mathscr{T}_w a(t) + \sum_j n_j(t)e_j\Big)b^w(t)\,\overline{s}_i \\
&= -\frac{d}{dt}\beta(t)f_i b^{ws_i}(t) \\
&\quad + \Big(\frac{d}{dt}\mathscr{T}_w a(t) + \sum_j n_j(t)e_j + n_i(t)\beta(t)h_i + n_i(t)\beta^2(t)f_i\Big)b^{ws_i}(t) \\
&= \Big[\Big(\frac{d}{dt}\beta(t) + \frac{d}{dt}\alpha_i(\mathscr{T}_w a(t)) + n_i(t)\beta^2(t)\Big)f_i \\
&\quad + \frac{d}{dt}\mathscr{T}_w a(t) + n_i(t)\beta(t)h_i + \sum_j n_j(t)e_j\Big]b^{ws_i}(t).
\end{aligned}$$

Since $b^{ws_i}(t) \in B_+$, one has $\frac{d}{dt}\beta(t) + \frac{d}{dt}\alpha_i(T_w a(t)) + \beta^2(t) = 0$. Therefore

$$\beta(t) = \frac{e^{-\alpha_i(\mathscr{T}_w a(t))}}{C + \int_0^t e^{-\alpha_i(T_w a(s))}n_i(s)\,ds}$$

for some constant $C \geq 0$. Integrating the H part of the Gauss decomposition of $b^{ws_i}(t)$, we see that this part is equal to

$$\exp\big(\mathscr{T}_w a(t)\big)\exp\bigg(C' + \log\Big(C + \int_0^t e^{-\alpha_i(T_w a(s))}n_i(s)\,ds\Big)\bigg)h_i. \tag{3.6}$$

Therefore

$$\frac{\Delta^{\omega_i}(b(t)\overline{ws}_i)}{\Delta^{\omega_i}(b(t)\overline{w})} = \exp(C')\Big(C + \int_0^t e^{-\alpha_i(\mathcal{T}_w a(s))} n_i(s)\,ds\Big)$$

and $C = 0$, by Lemma 3.7. We conclude that

$$\beta(t) = \frac{e^{-\alpha_i(\mathcal{T}_w a(t))}}{\int_0^t e^{-\alpha_i(T_w a(s))} n_i(s)\,ds}.$$

This implies that

$$\begin{aligned}\frac{d}{dt} b^{ws_i}(t) &= \Big[\frac{d}{dt}\mathcal{T}_w a(t) + n_i(t)\frac{e^{-\alpha_i(\mathcal{T}_w a(t))}}{\int_0^t e^{-\alpha_i(T_w a(s))} n_i(s)\,ds} h_i + \sum_j n_j(t) e_j\Big] b^{ws_i}(t)\\ &= \Big[\frac{d}{dt}\mathcal{T}_{i,n}\mathcal{T}_w a(t) + \sum_j n_j(t) e_j\Big] b^{ws_i}(t),\end{aligned}$$

as required. □

From (3.6), we obtain

$$\frac{\Delta^{\omega_i}(b(t)\overline{ws}_i)}{\Delta^{\omega_i}(b(t)\overline{w})} = \Delta^{\omega_i}\big(e^{-\mathcal{T}_w a(t)} b^w(t)\overline{s}_i\big) = \exp(C')\int_0^t e^{-\alpha_i(T_w a(s))} n_i(s)\,ds.$$

Differentiating with respect to t, we get

$$\begin{aligned}\frac{d}{dt} e^{-\mathcal{T}_w a(t)} b^w(t)\overline{s}_i &= e^{-\mathcal{T}_w a(t)} \sum_j n_j(t) e_j e^{\mathcal{T}_w a(t)} e^{-\mathcal{T}_w a(t)} b^w(t)\overline{s}_i\\ &= \Big(\sum_i e^{-\alpha_j(\mathcal{T}_w a(t))} n_j(t) e_j\Big) e^{-\mathcal{T}_w a(t)} b^w(t)\overline{s}_i,\end{aligned}$$

where $e^{-\mathcal{T}_w a(t)} b^w(t) \in N$. It follows that

$$\begin{aligned}\frac{d}{dt}\Big\{\frac{\Delta^{\omega_i}(b(t)\overline{ws}_i)}{\Delta^{\omega_i}(b(t)\overline{w})}\Big\} &= \Big\langle\Big(\sum_j e^{-\alpha_j(\mathcal{T}_w a(t))} n_j(t) e_j\Big) e^{-\mathcal{T}_w a(t)} b^w(t)\overline{s}_i v_{\omega_i}, v_{\omega_i}\Big\rangle\\ &= e^{-\alpha_i(\mathcal{T}_w a(t))} n_i(t)\langle e_i \overline{s}_i v_{\omega_i}, v_{\omega_i}\rangle\\ &= e^{-\alpha_i(\mathcal{T}_w a(t))} n_i(t).\end{aligned}$$

Therefore $C' = 0$. This proves the claim for $i = j$ and finishes the proof of Theorem 3.5. □

COROLLARY 3.9
The transformations $\mathcal{T}_{i,n}$ satisfy the braid relations

$$\underbrace{\mathcal{T}_{i,n}\mathcal{T}_{j,n}\cdots}_{m(i,\,j)\text{ terms}} = \underbrace{\mathcal{T}_{j,n}\mathcal{T}_{i,n}\cdots}_{m(i,\,j)\text{ terms}},$$

where $m(i, j)$ is the Cartan integer $\alpha_i(\alpha_j^\vee)$.

Remark 3.10
In the case of rank-two groups, the braid relations of Corollary 3.9 and an application of the Laplace method yield the braid relations for Pitman operators, as in Theorem 2.4, in the case of cristallographic angles $\pi/m, m = 2, 3, 4, 6$. It is instructive to give an elementary derivation of the braid relations for the $\mathcal{T}_{i,n}$ in the simplest nontrivial case, namely, type A_2 (i.e., $m = 3$). In this case, the relations amount to

$$\int_0^t ds \int_0^s dr\ F(r)\frac{G(s)}{G(r)}\frac{H(t)}{H(s)} = \int_0^t ds \int_0^s dr\ F(r)\frac{\tilde{G}(s)}{\tilde{G}(r)}\frac{\tilde{H}(t)}{\tilde{H}(s)} \tag{3.7}$$

for some positive continuous functions F, G, H, where

$$\tilde{G}(s) = \Big(\int_0^s G(r)H(r)^{-1}\,dr\Big)^{-1}G(s)$$

and

$$\tilde{H}(s) = \Big(\int_0^s G(r)H(r)^{-1}\,dr\Big)H(s).$$

This can be checked directly by an application of Fubini's theorem or by an integration by parts. Similar but more complicated formulas correspond to the other crystallographic angles $\pi/4$ and $\pi/6$.

From (3.7), one recovers, by the Laplace method, the identity

$$x \vartriangle (z \triangledown y) \vartriangle (y \vartriangle z) = (x \vartriangle y) \vartriangle z \tag{3.8}$$

for continuous functions x, y, z with $x(0) = y(0) = z(0) = 0$ and (nonassociative) binary operations $\triangledown$ and $\vartriangle$ defined by

$$(x \vartriangle y)(t) = \inf_{0\le s\le t}[x(s) - y(s) + y(t)], \tag{3.9}$$

$$(x \triangledown y)(t) = \sup_{0\le s\le t}[x(s) - y(s) + y(t)]. \tag{3.10}$$

This is equivalent to the $n = 3$ braid relation for the Pitman transforms. For a queuing-theoretic proof, which some readers might find illuminating, see [25]. Lemma 2.7 is a special case.

3.4. Representation-theoretic formula for $\mathcal{P}_w$

Let $w \in W$, and let λ be a dominant weight; then $\lambda - w\lambda$ can be decomposed as a linear combination of simple positive roots $\lambda - w\lambda = \sum_{i\in I} u_i\alpha_i$, where u_i are nonnegative integers. If $(j_1, \dots, j_r) \in I^r$ is a sequence such that $\langle e_{j_1} \cdots e_{j_r}\overline{w}v_\lambda, v_\lambda\rangle \neq 0$, then the number of k's in the sequence $j_1, \dots, j_r$ is equal to u_k. In particular, the number r depends only on w and λ. We let $S(\lambda, w)$ denote the set of sequences $(j_1, \dots, j_r) \in I^r$ such that $\langle e_{j_1} \cdots e_{j_r}\overline{w}v_\lambda, v_\lambda\rangle \neq 0$. Using (3.4) and (3.5), we obtain the following expression.

PROPOSITION 3.11
Let a be a path in $\mathfrak{a}$, and let λ be a dominant weight; then one has

$$\langle e^{\mathcal{T}_w a(t)}v_\lambda, v_\lambda\rangle = e^{\lambda(a(t))} \sum_{(j_1,\dots,j_r)\in S(\lambda,w)} \int_{t\ge t_1\ge\cdots\ge t_r\ge 0} e^{-\alpha_{j_1}(a(t_1))-\cdots-\alpha_{j_r}(a(t_r))} n_{j_1}(t_1) \cdots n_{j_r}(t_r)\, dt_1 \cdots dt_r \langle e_{j_1} \cdots e_{j_r}\overline{w}v_\lambda, v_\lambda\rangle.$$

Let $w \in W$, and let $\mathcal{P}_w^\vee$ denote the Pitman transformation on $\mathfrak{a}$ for the dual root system $R^\vee$. By (3.2), one has

$$\mathcal{P}_w^\vee \pi = \lim_{\varepsilon\to 0} D_\varepsilon \mathcal{T}_w D_\varepsilon^{-1}\pi.$$

Using the Laplace method, Lemma 3.1, and Proposition 3.11 applied to fundamental weights, we now obtain the following expression for the Pitman transform. (Notice that W acts on $\mathfrak{a}^*$ and $\mathfrak{a}$ by duality.)

THEOREM 3.12 (Representation-theoretic formula for the Pitman transforms)
Let $w \in W$. For each path π on $\mathfrak{a}$, one has

$$\mathcal{P}_w^\vee \pi(t) = \pi(t) - \sum_{i\in I} \inf_{\substack{j_1,\dots,j_r\in S(\omega_i,w)\\ t\ge t_1\ge t_2\cdots\ge t_r\ge 0}} \big(\alpha_{j_1}(\pi(t_1)) + \cdots + \alpha_{j_r}(\pi(t_r))\big)\alpha_i^\vee. \tag{3.11}$$

This formula can be seen as a generalization of the formula in Theorem 2.5. Observe that sequences $j_1, \dots, j_r$, such as the ones occurring in the theorem, have appeared already in [3] under the name of "**i**-trails." It is interesting to note that such sequences appear here naturally by an application of the Laplace method (sometimes called *tropicalization* in the algebraic literature).

By Corollary 2.10, we see that Theorem 3.12 provides a representation-theoretic formula for the dominant path in some Littelmann module which is independent of any choice of a reduced decomposition of w_0.

Remark 3.13
As noted before, formula (3.11) has a structure similar to formula (2.4) (when $\rho = \cos(\pi/n)$). We conjecture that such formulas exist for arbitrary Coxeter groups; that is, for $w \in W$, there exists r and sets $S(s, w) \subset S^r$ such that

$$\mathscr{P}_w\pi(t) = \pi(t) - \sum_{s\in S} \inf_{\substack{s_1,\dots,s_r\in S(s,w)\\ t\geq t_1\geq t_2\cdots\geq t_r\geq 0}} \big(\alpha_{s_1}^\vee(\pi(t_1)) + \cdots + \alpha_{s_r}^\vee(\pi(t_r))\big)\alpha_s. \tag{3.12}$$

However, we do not know how to interpret these sets $S(s, w)$.

4. Duality

4.1. An involution on dominant paths
As in Section 2.7, we consider a Coxeter system (W, S) generated by a set S of reflections of V. We assume now that the group W is finite, and we let w_0 be the longest element. We fix some $T > 0$; and for any continuous path $\pi : [0, T] \to V$ such that $\pi(0) = 0$, we let

$$\kappa\pi(t) = \pi(T - t) - \pi(T).$$

Clearly, for all paths, $\kappa^2\pi = \pi$. We show that the transformation $I = \mathscr{P}_{w_0}\kappa(-w_0)$ is an involution on the set of dominant paths which generalizes the Schützenberger involution (see Section 4.5 for the connection).

4.2. Codominant paths and co-Pitman operators
A path π is called *α-dominant* if $\alpha^\vee(\pi(t)) \geq 0$ for all t. It is called *α-codominant* if $\kappa\pi$ is α-dominant or, in other words, if $\alpha^\vee(\pi(t)) \geq \alpha^\vee(\pi(T))$ for all t. Finally, it is called *codominant* if it is α-codominant for all α. Let us define the *co-Pitman operators* $\mathscr{E}_\alpha = \kappa\mathscr{P}_\alpha\kappa$, which are given by the formula

$$\mathscr{E}_\alpha\pi(t) = \pi(t) - \inf_{t\leq s\leq T} \alpha^\vee\big(\pi(s)\big)\alpha + \inf_{0\leq s\leq T} \alpha^\vee\big(\pi(s)\big)\alpha.$$

One checks the following:

$$\mathscr{P}_\alpha\kappa\mathscr{P}_\alpha = \mathscr{P}_\alpha, \qquad \mathscr{E}_\alpha^2 = \mathscr{E}_\alpha, \qquad \mathscr{E}_\alpha\mathscr{P}_\alpha = \mathscr{E}_\alpha, \qquad \mathscr{P}_\alpha\mathscr{E}_\alpha = \mathscr{P}_\alpha.$$

Furthermore, for all paths π, one has

$$\mathscr{E}_\alpha \pi(T) = s_\alpha \mathscr{P}_\alpha \pi(T).$$

The proof of the following lemma is left to the reader.

LEMMA 4.1
The transformations $\mathscr{E}_\alpha$ satisfy the following properties.
(i) *$\mathscr{E}_\alpha \pi$ is the unique path η satisfying $\eta(T) = s_\alpha \mathscr{P}_\alpha \pi(T)$ and $\mathscr{P}_\alpha \eta = \mathscr{P}_\alpha \pi$.*
(ii) *$\mathscr{E}_\alpha \pi$ is the unique path η such that $\mathscr{P}_\alpha \eta = \mathscr{P}_\alpha \pi$ and that η is α-codominant.*
(iii) *If π is α-dominant, then $\mathscr{E}_\alpha \pi$ is the unique path such that $\mathscr{P}_\alpha \eta = \pi$ and $\eta(T) = s_\alpha(\pi(T))$.*
(iv) *$\mathscr{E}_\alpha \pi = \pi$ if and only if π is α-codominant.*

The transformations $\mathscr{E}_\alpha$ play the same role with respect to the Littelmann operators f_α as the transformations $\mathscr{P}_\alpha$ do with respect to e_α (see (2.3)).

LEMMA 4.2
The $\mathscr{E}_\alpha$ satisfy the braid relations.

Proof
The proof follows from $\mathscr{E}_\alpha = \kappa \mathscr{P}_\alpha \kappa$, $\kappa^2 = \mathrm{id}$, and the braid relations for the $\mathscr{P}_\alpha$. □

One can therefore define $\mathscr{E}_w$ for $w \in W$, and $\mathscr{E}_{w_0} = \mathscr{E}_{w_0}^2$ is a projection onto the set of codominant paths. Furthermore, for all $w \in W$, one has

$$\mathscr{E}_w = \kappa \mathscr{P}_w \kappa.$$

In particular,

$$\mathscr{E}_{w_0} = \kappa \mathscr{P}_{w_0} \kappa.$$

4.3. An endpoint property
In this section, we prove the following result, which is crucial for applications to Brownian motion.

PROPOSITION 4.3
For any path π, one has

$$\mathscr{E}_{w_0} \pi(T) = w_0 \mathscr{P}_{w_0} \pi(T).$$

Since $\mathscr{P}_{w_0} \mathscr{E}_{w_0} = \mathscr{P}_{w_0}$, it is enough to check this identity for π a codominant path (or for a dominant path using $\mathscr{E}_{w_0} \mathscr{P}_{w_0} = \mathscr{E}_{w_0}$).

LEMMA 4.4
Let π be a codominant path, let $w \in W$, and let α be such that $l(s_\alpha w) > l(w)$; then $\mathscr{P}_w \pi$ is α-codominant.

Proof
First we check the result for dihedral groups. With the notation of Section 2.4, let π be an α- and β-codominant path, and let n be such that $\rho > \cos(\pi/n)$; then one has $\alpha^\vee(\pi(T)) \le \alpha^\vee(\pi(t))$ and $\beta^\vee(\pi(T)) \le \beta^\vee(\pi(t))$ for all $t \le T$. It follows that in the computation of $\underbrace{\mathscr{P}_\beta\mathscr{P}_\alpha\mathscr{P}_\beta\cdots}_{n\text{ terms}}\pi(T)$ using formula (2.4), the infimum is obtained for $s_0 = s_1 = \cdots = T$; therefore (assuming n odd for definiteness)

$$\begin{aligned}
&\alpha^\vee\big(\mathscr{P}_\beta\mathscr{P}_\alpha\mathscr{P}_\beta\cdots\pi(T)\big)\\
&\quad= \alpha^\vee\big(\pi(T)\big) + 2\rho\big[\beta^\vee(\pi(T)) + T_1(\rho)\alpha^\vee(\pi(T)) + \cdots + T_{n-1}(\rho)\beta^\vee(\pi(T))\big]\\
&\qquad -2\big[\alpha^\vee(\pi(T)) + T_1(\rho)\beta^\vee(\pi(T)) + \cdots + T_{n-2}(\rho)\beta^\vee(\pi(T))\big]\\
&\quad= T_{n-1}(\rho)\alpha^\vee\big(\pi(T)\big) + T_n(\rho)\beta^\vee\big(\pi(T)\big),
\end{aligned}$$

where we have used the recursion relation of the T_k. On the other hand, for $t \le T$, one has

$$\begin{aligned}
&\alpha^\vee\big(\mathscr{P}_\beta\mathscr{P}_\alpha\mathscr{P}_\beta\ldots\pi(t)\big)\\
&\quad= \alpha^\vee\big(\pi(t)\big) + 2\rho \inf_{t\ge s_0\ge\cdots\ge s_{n-1}\ge 0}\big[\beta^\vee(\pi(s_0)) + T_1(\rho)\alpha^\vee(\pi(s_1))\\
&\qquad\qquad + \cdots + T_{n-1}(\rho)\beta^\vee(\pi(s_{n-1}))\big]\\
&\qquad -2\inf_{t\ge s_0\ge\cdots\ge s_{n-2}\ge 0}\big[\alpha^\vee(\pi(s_0)) + T_1(\rho)\beta^\vee(\pi(s_1))\\
&\qquad + \cdots + T_{n-2}(\rho)\beta^\vee(\pi(s_{n-2}))\big].
\end{aligned}$$

In this expression, inside the $\inf_{t\ge s_0\ge s_1\ge\cdots\ge s_{n-1}\ge 0}$, let us replace each $2\rho T_k(\rho)$ by $T_{k-1}(\rho) + T_{k+1}(\rho)$. We obtain

$$\begin{aligned}
&\inf_{t\ge s_0\ge s_1\ge\cdots\ge s_{n-1}\ge 0}\big[2\rho\beta^\vee(\pi(s_0)) + (T_0(\rho)+T_2(\rho))\alpha^\vee(\pi(s_1))\\
&\qquad\qquad + \cdots + (T_{n-2}(\rho)+T_n(\rho))\beta^\vee(\pi(s_{n-1}))\big]\\
&\quad\ge \inf_{\substack{t\ge s_0\ge s_1\ge\cdots\ge s_{n-1}\ge 0\\ t\ge u_1\ge\cdots\ge u_{n-1}\ge 0}}\big[2\rho\beta^\vee(\pi(s_0)) + T_2(\rho)\alpha^\vee(\pi(s_1))\\
&\qquad\qquad + \cdots + T_n(\rho)\beta^\vee(\pi(s_{n-1})) + T_0(\rho)\alpha^\vee(\pi(u_1))\\
&\qquad\qquad + \cdots + T_{n-2}(\rho)\beta^\vee(\pi(u_{n-1}))\big]\\
&\quad= \inf_{t\ge s_0\ge\cdots\ge s_{n-1}\ge 0}\big[2\rho\beta^\vee(\pi(s_0)) + T_2(\rho)\alpha^\vee(\pi(s_1)) + \cdots + T_n(\rho)\beta^\vee(\pi(s_{n-1}))\big]\\
&\qquad + \inf_{t\ge s_0\ge\cdots\ge s_{n-2}\ge 0}\big[\alpha^\vee(\pi(s_0)) + T_1(\rho)\beta^\vee(\pi(s_1)) + \cdots + T_{n-2}(\rho)\beta^\vee(\pi(s_{n-2}))\big].
\end{aligned}$$

Furthermore,

$$\alpha^\vee\big(\pi(t)\big) + \inf_{t\geq s_0\geq\cdots\geq s_{n-1}\geq 0}\big[2\rho\beta^\vee(\pi(s_0)) + T_2(\rho)\alpha^\vee(\pi(s_1)) + \cdots + T_n(\rho)\beta^\vee(\pi(s_{n-1}))\big]$$
$$\geq \inf_{t\geq s_0\geq\cdots\geq s_{n-2}\geq 0}\big[\alpha^\vee(\pi(s_0)) + T_1(\rho)\beta^\vee(\pi(s_1)) + \cdots + T_{n-2}(\rho)\beta^\vee(\pi(s_{n-2}))\big]$$
$$+ T_{n-1}(\rho)\alpha^\vee\big(\pi(T)\big) + T_n(\rho)\beta^\vee\big(\pi(T)\big).$$

Putting everything together, we obtain

$$\alpha^\vee\big(\mathscr{P}_\beta\mathscr{P}_\alpha\mathscr{P}_\beta\cdots\pi(t)\big) \geq \alpha^\vee\big(\mathscr{P}_\beta\mathscr{P}_\alpha\mathscr{P}_\beta\ldots\pi(T)\big),$$

and $\mathscr{P}_\beta\mathscr{P}_\alpha\mathscr{P}_\beta\cdots\pi$ is α-codominant. The case of n even is similar. This proves the claim for dihedral groups.

Consider now a general Coxeter system. We do the proof by induction on $l(w)$. The claim is true if $l(w) = 0$. If it is true for some w, let $s_\beta \in S$ be such that $l(s_\beta w) > l(w)$. Now let α be such that $l(s_\alpha s_\beta w) > l(s_\beta w) > l(w)$. Let n be the order of $s_\alpha s_\beta$, and let

$$w = s_\alpha w_1 = s_\alpha s_\beta w_2 = s_\alpha s_\beta s_\alpha w_3 = \cdots = s_\alpha s_\beta\cdots w_k,$$

where k is the smallest integer such that

$$l(w) > l(w_1) > \cdots > l(w_k)$$

and

$$l(s_\alpha w_k) > l(w_k), \qquad l(s_\beta w_k) > l(w_k).$$

Since $l(s_\alpha s_\beta w) = l(w_k) + k + 2$, one has $k + 2 \leq n$. By the induction hypothesis, $\mathscr{P}_{w_k}(\pi)$ is both α- and β-codominant. Then it follows from the dihedral case that $\mathscr{P}_{s_\beta}\mathscr{P}_w = \mathscr{P}_\beta\mathscr{P}_w\pi = \mathscr{P}_\beta\mathscr{P}_\alpha\mathscr{P}_\beta\cdots\mathscr{P}_{w_k}\pi$ is α-codominant. □

LEMMA 4.5

Let π be a codominant path, and let $w \in W$; then $\mathscr{P}_w\pi$ is the unique path η such that $\mathscr{E}_{w^{-1}}\eta = \pi$ and $w(\pi(T)) = \eta(T)$.

Proof

The proof is by induction on $l(w)$, using Lemma 4.4. Let $l(s_\alpha w) = l(w) + 1$; then $\mathscr{P}_w\pi$ is α-codominant. Therefore $\mathscr{P}_\alpha\mathscr{P}_w\pi$ is the unique path η such that $\mathscr{E}_\alpha\eta = \mathscr{P}_w\pi$ and $\eta(T) = s_\alpha\mathscr{P}_w\pi(T)$. □

Proposition 4.3 is the special case $w = w_0$ in Lemma 4.5.

LEMMA 4.6
We have

$$(-w_0)\mathscr{P}_{w_0} = \mathscr{P}_{w_0}(-w_0).$$

Proof
If α is a simple root, then $\tilde{\alpha} = -w_0\alpha$ is also a simple root, and $\tilde{\alpha}^\vee = -\alpha^\vee w_0$. It follows easily that $(-w_0)\mathscr{P}_\alpha(-w_0) = \mathscr{P}_{\tilde{\alpha}}$.

If $w_0 = \alpha_1 \cdots \alpha_r$ is a reduced expression, we thus have

$$\mathscr{P}_{w_0}(-w_0) = \mathscr{P}_{\alpha_1} \cdots \mathscr{P}_{\alpha_r}(-w_0) = (-w_0)\mathscr{P}_{\tilde{\alpha}_1} \cdots \mathscr{P}_{\tilde{\alpha}_r} = (-w_0)\mathscr{P}_{w_0}$$

since $w_0 = \tilde{\alpha}_1 \cdots \tilde{\alpha}_r$. □

THEOREM 4.7
The transformation $I = \mathscr{P}_{w_0}(-w_0)\kappa$ has the following properties:
(i) $I^2 = \mathscr{P}_{w_0}$;
(ii) *the restriction of I to dominant paths is an involution;*
(iii) $I\mathscr{P}_{w_0} = I$;
(iv) *the duality relation: for all paths π, one has*

$$I\pi(T) = \mathscr{P}_{w_0}\pi(T).$$

In particular, one has $I\pi(T) = \pi(T)$ when π is dominant.

Proof
By Lemma 4.6,

$$I^2 = \mathscr{P}_{w_0}\kappa(-w_0)\mathscr{P}_{w_0}(-w_0)\kappa = \mathscr{P}_{w_0}\kappa\mathscr{P}_{w_0}\kappa = \mathscr{P}_{w_0}\mathscr{E}_{w_0} = \mathscr{P}_{w_0}.$$

This proves (i) and implies (ii) since $\mathscr{P}_{w_0}\pi = \pi$ when π is dominant. This also gives

$$I\mathscr{P}_{w_0} = I^3 = I^2 I = I$$

since the image by I of any path is dominant. Finally, $I = \mathscr{P}_{w_0}\kappa(-w_0) = \kappa\mathscr{E}_{w_0}(-w_0)$, and Proposition 4.3 gives (iv). □

Property (iv) is important for the first proof of the Brownian motion property.

4.4. Symmetry of a Littlewood-Richardson construction
The concatenation $\pi \star \eta$ of two paths $\pi : [0, T] \to V$ $\eta : [0, T] \to V$ is defined in Littelmann [22] as the path $\pi \star \eta : [0, T] \to V$ given by $\pi \star \eta(t) = \pi(2t)$ when $0 \le t \le T/2$ and $\pi \star \eta(t) = \pi(T) + \eta(2(t - T/2))$ when $T/2 \le t \le T$.

LEMMA 4.8
For all $w \in W$, one has $\mathscr{P}_w(\pi \star \eta) = \mathscr{P}_w(\pi) \star \eta'$, where $\mathscr{P}_{w_0}(\eta') = \mathscr{P}_{w_0}(\eta)$.

Proof
One uses induction on the length $l(w)$ of w. When $l(w) = 1$, it is easy to see that $\mathscr{P}_w(\pi \star \eta) = \mathscr{P}_w(\pi) \star \eta'$, where $\mathscr{P}_w(\eta) = \mathscr{P}_w(\eta')$. Since $\mathscr{P}_{w_0}\mathscr{P}_w = \mathscr{P}_{w_0}$, the claim is thus true in this case. Suppose that it holds for elements of length n. Let $w = w_1 s$, where $l(w) = n + 1$, $l(w_1) = n$; then one has

$$\mathscr{P}_w(\pi \star \eta) = \mathscr{P}_{w_1}\mathscr{P}_s(\pi \star \eta) = \mathscr{P}_{w_1}\big(\mathscr{P}_s(\pi) \star \eta'\big),$$

where $\mathscr{P}_{w_0}\eta' = \mathscr{P}_{w_0}\eta$. Now, by induction hypothesis,

$$\mathscr{P}_{w_1}\big(\mathscr{P}_s(\pi) \star \eta'\big) = (\mathscr{P}_{w_1}\mathscr{P}_s)(\pi) \star \eta'',$$

where $\mathscr{P}_{w_0}\eta'' = \mathscr{P}_{w_0}\eta'$; and therefore $\mathscr{P}_{w_0}\eta'' = \mathscr{P}_{w_0}\eta$. □

In the case of Weyl groups, Littelmann has given the following analogue of the Littlewood-Richardson construction. Let π and η be two integral dominant paths defined on $[0, T]$; then the set

$$\mathrm{LR}(\pi, \eta) = \{\pi \star \mu \mid \mu \in B\eta, \pi \star \mu \text{ is dominant}\}$$

gives a parametrization of the decomposition into irreducible representations of the tensor product of the representations with highest weights $\pi(T)$ and $\eta(T)$. By Theorem 4.7(iii), one has $I(\eta)(T) = \eta(T)$ and $I(\pi)(T) = \pi(T)$; therefore $\mathrm{LR}(I(\eta), I(\pi))$ gives a parametrization of the decomposition of the tensor product of the representations with highest weights $\eta(T)$ and $\pi(T)$.

PROPOSITION 4.9
The map $I : \mathrm{LR}(\pi, \eta) \to \mathrm{LR}(I(\eta), I(\pi))$ is a bijective involution, which preserves the endpoints.

Proof
Let $\pi \star \mu \in \mathrm{LR}(\pi, \eta)$. By Lemma 4.8, there is a path ξ such that

$$\begin{aligned} I(\pi \star \mu) &= \mathscr{P}_{w_0}\big(\kappa(-w_0)(\pi \star \mu)\big) = \mathscr{P}_{w_0}\big(\kappa(-w_0)(\mu) \star \kappa(-w_0)(\pi)\big) \\ &= \mathscr{P}_{w_0}\big(\kappa(-w_0)(\mu)\big) \star \xi \end{aligned}$$

and $\mathscr{P}_{w_0}\xi = \mathscr{P}_{w_0}(\kappa(-w_0)(\pi)) = I(\pi)$. By Theorem 4.7(iii), one has $I(\mu) = I(\eta)$; thus $I(\pi \star \eta) \in \mathrm{LR}(I(\eta), I(\pi))$. One easily checks that I preserves integrality, and the other properties follow from Theorem 4.7. □

4.5. Connection with the Schützenberger involution

In the case of a Weyl group of type A_{d-1}, the transform $\mathscr{P}_{w_0}$ is connected with the Robinson, Schensted, and Knuth (RSK) correspondence. Let us consider a word $v_1 v_2 \cdots v_n$ written with the alphabet $\{1, 2, \ldots, d\}$. Let $(P(n), Q(n))$ be the pair of tableaux associated with this word by RSK with column insertion (see, e.g., [14]). Let $\mathfrak{a} = \big\{(x_1, \ldots, x_d) \in \mathbb{R}^d; \sum_{i=1}^d x_i = 0\big\}$, and let (e_i) be the image in $\mathfrak{a}$ of the canonical basis of $\mathbb{R}^d$. We identify v_i with the path $\eta_i : t \mapsto te_{v_i}$, $0 \le t \le 1$, and we consider the path $\pi = \eta_1 \star \eta_2 \cdots \star \eta_n$. Then $\mathscr{P}_{w_0}\pi$ is the path obtained by taking the successive shapes of $Q(1), Q(2), \ldots, Q(n)$ (see Littelmann [22], [24], or [25] for a direct combinatorial proof). Let us consider the pair $(\tilde{P}(n), \tilde{Q}(n))$ associated by the RSK algorithm to the word $v_n^* \cdots v_1^*$, where $v^* = d + 1 - v$. The Schützenberger involution is the map that associates the tableau $\tilde{Q}(n)$ to the tableau $Q(n)$ (see [12], [14], [21]). The path associated with the word $v_n^* \cdots v_1^*$ is $I(\pi)$. Thus I is a generalization of this involution. Note that I makes sense not only for Weyl groups but also for any finite Coxeter group.

5. Representation of Brownian motion in a Weyl chamber

5.1. Brownian motion in a Weyl chamber

In this section, we recall some basic facts about Brownian motion in Weyl chambers.

We consider a Coxeter system (W, S) generated by a set S of reflections of a Euclidean space V, and we assume that W is finite. We denote by C the interior of a fundamental domain for the action of W on V (a Weyl chamber), and we denote by $\overline{C}$ its closure.

If W is the Weyl group of a complex semisimple Lie algebra $\mathfrak{g}$ with compact form $\mathfrak{g}_{\mathbb{R}}$, then V is identified with $\mathfrak{a}^*$, the dual space of the Lie algebra of a maximal torus T, and the Weyl chamber $\overline{C} = \overline{\mathfrak{a}}_+^*$ can be identified with the orbit space of $\mathfrak{g}_{\mathbb{R}}^*$ under the coadjoint action of the simply connected compact group K with Lie algebra $\mathfrak{g}_{\mathbb{R}}$ (up to some identification of the walls). Let Z be a Brownian motion with values in $\mathfrak{g}_{\mathbb{R}}^*$ whose covariance is the Killing form. It is well known that the image of Z in the quotient space $\mathfrak{g}_{\mathbb{R}}^*/K$ remains in the interior of the Weyl chamber for all times $t > 0$, even if the starting point is inside some wall. Since the transition probabilities of Z are invariant under the coadjoint action, it follows that this image, under the quotient map, is a Markov process on $\overline{C}$. A description of this Markov process can be done in terms of Doob's conditioning. Namely, the process is obtained from a Brownian motion X on $V = \mathfrak{a}^*$ killed at the boundary of the Weyl chamber by means of a Doob transform with respect to the function

$$h(v) = \prod_{\alpha \in R^+} \alpha^\vee(v), \quad v \in V,$$

where R^+ is the set of positive roots, which is the unique, up to a scaling factor, positive harmonic function on $\overline{C}$ which vanishes on the boundary (see [6]). Recall that, by the reflection principle, the transition probabilities for the Brownian motion killed at the boundary of the Weyl chamber are

$$p_t^0(x, y)\,dy = \sum_{w \in W} \varepsilon(w) p_t(x, wy)\,dy, \quad x, y \in \overline{C}, \tag{5.1}$$

where $p_t(x, y)\,dy$ are the transition probabilities for the Brownian motion X given by the Gaussian kernel on $\mathfrak{a}^*$ whose covariance is that of the Brownian motion. Thus the probability transitions for the Doob process are

$$q_t(x, y)\,dy = \frac{h(y)}{h(x)} \sum_{w \in W} \varepsilon(w) p_t(x, wy)\,dy \tag{5.2}$$

for $x \in C$. These probability transitions can be continued by continuity to $x \in \overline{C}$, in particular, to $x = 0$.

For a general finite Coxeter group, formula (5.1) still gives the probability transitions of the Brownian motion killed at the boundary of the Weyl chamber. Let h be the product of the positive coroots, defined as the linear forms corresponding to the hyperplanes of the reflections in the group W, taking the signs so that they are positive inside the Weyl chamber; then the function h is still the only (up to a multiplicative constant) positive harmonic function vanishing on the boundary, and the equation (5.2) defines the semigroup of what we call the Brownian motion in the fundamental chamber $\overline{C}$ of V.

We prove that the Pitman operator $\mathscr{P}_{w_0}$ applied to Brownian motion in V yields a Brownian motion in the Weyl chamber. We give two very different proofs of this. The first one uses in an essential way the duality relation of Proposition 4.3 and a classical result in queuing theory. The second one uses a random walk approximation and relies on Littelmann theory and the Weyl character formula. It is valid only for Weyl groups. We have chosen to present this second proof because it emphasizes the close connection between Brownian paths and Littelmann paths.

5.2. Brownian motion with a drift

We now consider a Brownian motion in V with invariant covariance but also with a drift $\xi \in C$. Its transition probabilities are now

$$p_{t,\xi}(x, y) = p_t(x, y) \exp\left(\langle \xi, y - x\rangle - \frac{\|\xi\|^2}{2} t\right).$$

Actually, the distribution of this Brownian motion on the σ-field $\mathscr{F}_t$ generated by the coordinate functions X_s, $s \leq t$, on the canonical space is absolutely continuous with respect to the distribution of the centered Brownian motion with density

$$\exp\Big(\langle \xi, X_t - X_0\rangle - \frac{\|\xi\|^2}{2}t\Big).$$

Consider such a Brownian motion in V with drift ξ, starting inside the chamber at point x, and killed at the boundary of C. The distribution of this process at time t is therefore given by the density, for $y \in C$,

$$\begin{aligned} p_t^0(x, y)\exp\Big(\langle \xi, y-x\rangle - \frac{\|\xi\|^2}{2}t\Big) &= \sum_{w\in W} \varepsilon(w)p_t(x, wy)\exp\Big(\langle \xi, y-x\rangle - \frac{\|\xi\|^2}{2}t\Big) \\ &= \sum_{w\in W} \varepsilon(w)p_t(0, y-wx)\exp\Big(\langle \xi, y-x\rangle - \frac{\|\xi\|^2}{2}t\Big), \end{aligned}$$

where we have used the invariance of p_t under the Weyl group. We now integrate this density over C in order to get the probability that the exit time from C is larger than t. Denoting by T_C this exit time, one has

$$P(T_C > t) = \sum_{w\in W} \varepsilon(w) \int_C p_t(0, y - wx)\exp\Big(\langle \xi, y - x\rangle - \frac{\|\xi\|^2}{2}t\Big)\, dy.$$

Since the drift ξ is in the chamber, for large t one has

$$\int_{V\setminus C} \exp\Big(\langle \xi, y - x\rangle - \frac{\|\xi\|^2}{2}t\Big)\, dy \to 0.$$

Therefore

$$\int_C p_t(0, y - wx)\exp\Big(\langle \xi, y - x\rangle - \frac{\|\xi\|^2}{2}t\Big)\, dy \underset{t\to\infty}{\longrightarrow} \exp\big(\langle \xi, w(x) - x\rangle\big)$$

and

$$\lim_{t\to\infty} P(T_C > t) = P(T_C = \infty) = \sum_{w\in W} \varepsilon(w)\exp\big(\langle \xi, w(x) - x\rangle\big).$$

We denote this function by $h_\xi(x)$. It follows that, conditionally on $\{T_C = \infty\}$, the Brownian motion with drift ξ, starting in C and killed at the boundary of C, is a Markov process with transition probabilities

$$q_{t,\xi}(x, y) = p_t^0(x, y)\frac{h_\xi(y)}{h_\xi(x)}\exp\Big(\langle \xi, y - x\rangle - \frac{\|\xi\|^2}{2}t\Big).$$

Observe that $h_\xi(y)/h_\xi(x) \to h(y)/h(x)$ as $\xi \to 0$. Standard arguments now show that as $x \to 0$ and $\xi \to 0$, the distribution of this process approaches that of the Brownian motion in the Weyl chamber, starting from zero.

Finally, we can rephrase this in the following way.

LEMMA 5.1
The distribution of the Brownian motion with a drift $\xi \in C$, started at zero and conditioned to stay forever in the cone $C - x$ (where $x \in C$), converges toward the distribution of the Brownian motion in the Weyl chamber when $x, \xi \to 0$.

5.3. Some further path transformations
Let $w_0 = s_1 \cdots s_q$ be a reduced decomposition, and write $\alpha_i = \alpha_{s_i}$. Let $\eta : [0, T] \to V$ be a path with $\eta(0) = 0$. Recall that η is dominant if $\eta(t) \in \overline{C}$ for all $t \leq T$. Set $\eta_q = \eta$, and for $j = 1, \ldots, q$, let

$$\eta_{j-1} = \mathscr{P}_{s_j} \cdots \mathscr{P}_{s_q} \eta_q, \qquad x_j = -\inf_{0 \leq t \leq T} \alpha_j^\vee\big(\eta_j(t)\big).$$

Then

$$\mathscr{P}_{w_0} \eta(T) = \eta(T) + \sum_{j=1}^{q} x_j \alpha_j,$$

and η is dominant if and only if $x_j = 0$ for all $j \leq q$. We now introduce some new path transformations and give an alternative characterization of dominant paths.

Let $w \in W$ be a reflection; that is, w is conjugate to some element in S. We choose a nonzero element α of V such that $w\alpha = -\alpha$; then w is the reflection s_α given by (2.1), where $\alpha^\vee(v) = 2(\alpha, v)/(\alpha, \alpha)$. As in [18], we call α a positive root when $\alpha^\vee$ is positive on the Weyl chamber C; it is a simple root when $s_\alpha \in S$. Observe that one has $\mathscr{P}_\alpha = \mathscr{P}_{s_\alpha}$ for all positive roots. (The left-hand side is defined by Definition 2.1, and the right-hand side is defined by Matsumoto's lemma in [8] since $s_\alpha \in W$.)

Let β be a positive root, and let s_β be the associated reflection. For any positive root α, one has

$$s_\beta \, \mathscr{P}_\alpha \, s_\beta = \mathscr{P}_{s_\beta(\alpha)}.$$

Consider the transformation $\mathscr{Q}_\beta = \mathscr{P}_\beta \, s_\beta$. One has

$$\mathscr{Q}_\beta \, \eta(t) = s_\beta \, \eta(t) + \sup_{0 \leq s \leq t} \beta^\vee\big(\eta(s)\big) \, \beta.$$

Furthermore, if $w_0 = s_1 \cdots s_q$ is a reduced decomposition ($s_i \in S$), then

$$\mathscr{Q}_{w_0} := \mathscr{P}_{w_0} \, w_0 = \mathscr{Q}_{\beta_1} \, \cdots \, \mathscr{Q}_{\beta_q},$$

where $\beta_1 = \alpha_1$ and $\beta_j = s_1 \cdots s_{j-1} \alpha_j$.

Now define transformations $\mathscr{D}_\alpha = s_\alpha \mathscr{E}_\alpha = \iota\, \mathscr{Q}_\alpha\, \iota$, where $\iota = -\kappa$. One has

$$\mathscr{D}_\alpha \eta(t) = \eta(t) + \inf_{T \geq u \geq t} \alpha^\vee\big(\eta(u) - \eta(t)\big)\alpha - \inf_{T \geq u \geq 0} \alpha^\vee\big(\eta(u)\big)\alpha. \tag{5.3}$$

Set

$$\mathscr{D}_{w_0} = \mathscr{D}_{\beta_1} \cdots \mathscr{D}_{\beta_q} = \iota\, \mathscr{Q}_{w_0}\, \iota,$$

and note that $\mathscr{D}_{w_0} = \iota\, \mathscr{P}_{w_0}\,(-\kappa)\, w_0$.

For a path η, set $\rho_q = \eta$, and for $j \leq q$, let

$$\rho_{j-1} = \mathscr{D}_{\beta_j} \cdots \mathscr{D}_{\beta_q}\, \rho_q, \qquad y_j = -\inf_{T \geq u \geq 0} \beta_j^\vee\big(\rho_j(u)\big).$$

LEMMA 5.2
For all paths η, one has

$$\mathscr{P}_{w_0}\eta(T) = \eta(T) + \sum_{j=1}^{q} y_j \beta_j. \tag{5.4}$$

In particular, η is dominant if and only if $y_j = 0$ for all $j \leq q$.

Proof
By construction,

$$\mathscr{D}_{w_0}\eta(T) = \eta(T) + \sum_{j=1}^{q} y_j \beta_j.$$

Since $\mathscr{D}_{w_0}\eta(T) = \mathscr{P}_{w_0}\eta(T)$ by Proposition 4.3, this implies (5.4). The path η is dominant if and only if $\mathscr{P}_{w_0}\eta(T) = \eta(T)$. By (5.4), this holds if and only if $\sum_j y_j \beta_j = 0$, and since the y_j and β_j are all positive, this is equivalent to the statement that $y_j = 0$ for all $j \leq q$. □

5.4. The representation theorem, first proof
The definitions of transformations $\mathscr{P}_\alpha$, $\mathscr{P}_{w_0}$, $\mathscr{Q}_\alpha$, $\mathscr{Q}_{w_0}$ extend naturally to paths π defined on $\mathbb{R}^+$. In this section, we prove that if X is a Brownian motion in V (started from the origin), then $\mathscr{Q}_{w_0}X$ is a Brownian motion in the fundamental chamber $\overline{C}$. Since w_0 leaves the distribution of Brownian motion invariant, this implies that $\mathscr{P}_{w_0}X$ is a Brownian motion in $\overline{C}$.

To prove this, we first extend the definition of the $\mathscr{D}_\beta$. Let β be a positive root. For paths $\pi : [0, +\infty) \to V$ with $\pi(0) = 0$ and $\alpha^\vee(\pi(t)) \to +\infty$ as $t \to +\infty$ for all simple roots α, define

$$\mathscr{D}_\beta\pi(t) = \pi(t) + \inf_{s\geq t}\beta^\vee\big(\pi(s)-\pi(t)\big)\beta - \inf_{s\geq 0}\beta^\vee\big(\pi(s)\big)\beta. \tag{5.5}$$

Now set $\mathscr{D}_{w_0} = \mathscr{D}_{\beta_1}\cdots\mathscr{D}_{\beta_q}$, as before. Since $\mathscr{D}_{w_0}$ does not depend on the chosen reduced decomposition of w_0, we can also write $\mathscr{D}_{w_0} = \mathscr{D}_{\beta_q}\cdots\mathscr{D}_{\beta_1}$.

LEMMA 5.3
If π is a dominant path, one has $\mathscr{Q}_{w_0}\mathscr{D}_{w_0}\pi = \pi$.

Proof
It is easy to see that for any positive root β and path $\xi : [0,\infty) \to V$ with $\xi(0) = 0$ and $\inf_{t\geq 0}\beta^\vee(\xi(t)) = 0$, we have $\mathscr{Q}_\beta\mathscr{D}_\beta\xi = \xi$. Let $\eta_0 = \pi$, and let

$$\eta_j = \mathscr{D}_{\beta_j}\cdots\mathscr{D}_{\beta_1}\pi, \qquad v_j(t) := -\inf_{u\geq t}\beta_j^\vee\big(\eta_{j-1}(u)-\eta_{j-1}(t)\big).$$

Since π is dominant, we have, by Lemma 5.2 (with $T\to\infty$), $v_j(0) = 0$ for each $j \leq q$, and hence

$$\mathscr{Q}_{w_0}\mathscr{D}_{w_0}\pi = \mathscr{Q}_{\beta_1}\cdots\mathscr{Q}_{\beta_q}\mathscr{D}_{\beta_q}\cdots\mathscr{D}_{\beta_1}\pi = \pi,$$

as required. □

LEMMA 5.4
If X is a Brownian motion with drift in C, then $\mathscr{D}_{w_0}X$ has the same distribution as X and, moreover, is independent of the collection of random variables $\{\inf_{t\geq 0}\alpha^\vee(X(t)),\ \alpha$ simple root$\}$.

Proof
To prove this, we first need to extend the definitions of $\mathscr{D}_\beta$ and $\mathscr{Q}_\beta$ to paths π defined on $\mathbb{R}$ with $\pi(0) = 0$ and $\alpha^\vee(\pi(t)) \to \pm\infty$ as $t \to \pm\infty$ for all simple α. For $t \in \mathbb{R}$, set

$$\mathscr{Q}_\beta\pi(t) = s_\beta\,\pi(t) + \sup_{s\leq t}\beta^\vee\big(\pi(s)\big)\,\beta - \sup_{s\leq 0}\beta^\vee\big(\pi(s)\big)\,\beta,$$

and define $\mathscr{D}_\beta\pi$ by (5.5), allowing $t \in \mathbb{R}$. Then, if ι denotes the involution

$$\iota\,\pi(t) = -\pi(-t),$$

one has $\mathscr{D}_\beta = \iota\,\mathscr{Q}_\beta\,\iota$ and $\mathscr{D}_{w_0} := \mathscr{D}_{\beta_q}\cdots\mathscr{D}_{\beta_1} = \iota\,\mathscr{Q}_{w_0}\,\iota$, as before. Note that $\mathscr{D}_{w_0}$ does not depend on the particular reduced decomposition of w_0 and also that $\mathscr{D}_\beta(\pi(t), t\geq 0) = (\mathscr{D}_\beta\pi(t), t\geq 0)$ and $\mathscr{D}_{w_0}(\pi(t), t\geq 0) = (\mathscr{D}_{w_0}\pi(t), t\geq 0)$. We use the following auxiliary lemma.

LEMMA 5.5
Let $\pi : \mathbb{R} \to V$ with $\pi(0) = 0$, and let $\alpha(\pi(t)) \to \pm\infty$ as $t \to \pm\infty$ for all simple roots α. Then, for all $t \in \mathbb{R}$,

$$-\inf_{u\geq t} \beta^\vee\big(\pi(u) - \pi(t)\big) = -\inf_{s\leq t} \beta^\vee\big(\mathscr{D}_\beta\, \pi(u) - \mathscr{D}_\beta\, \pi(t)\big).$$

Proof
This can be checked directly or deduced from (2.2). □

Introduce a Brownian motion Y indexed by $\mathbb{R}$ such that $X = (Y(t), t \geq 0)$ and $(\iota Y(t), t \geq 0)$ is an independent copy of X. For any positive root β, the distribution of $\mathscr{D}_\beta Y$ is the same as that of Y. This is a one-dimensional statement that can be checked directly or can be seen as a consequence of the classical output theorem on the $(M/M/1)$-queue (see, e.g., [26]). In particular, the distribution of $\mathscr{D}_\beta X$ is the same as that of X. It follows that $\mathscr{D}_{w_0} Y$ has the same distribution as Y and that $\mathscr{D}_{w_0} X$ has the same distribution as X. Let $Y_0 = Y$, and let

$$Y_j = \mathscr{D}_{\beta_j} \cdots \mathscr{D}_{\beta_1} Y, \qquad V_j(t) := -\inf_{u\geq t} \beta_j^\vee\big(Y_{j-1}(u) - Y_{j-1}(t)\big).$$

Note that $Y_q = \mathscr{D}_{w_0} Y$, and recall that, for $t \geq 0$, $\mathscr{D}_{w_0} Y(t) = \mathscr{D}_{w_0} X(t)$. By Lemma 5.5, one has

$$V_q(t) = -\inf_{s\leq t} \beta_j^\vee\big(Y_q(s) - Y_q(t)\big),$$

$$Y_{q-1}(t) = Y_q(t) + \big(V_q(t) - V_q(0)\big)\beta_q$$

and, by induction on k,

$$V_{q-k}(t) = -\inf_{s\leq t} \beta_j^\vee\big(Y_{q-k}(s) - Y_{q-k}(t)\big),$$

$$Y_{q-k-1}(t) = Y_{q-k}(t) + \big(V_{q-k}(t) - V_{q-k}(0)\big)\beta_{q-k}.$$

It follows that the $(V_j(t), t \leq 0)$ are measurable with respect to the σ-field generated by $(\mathscr{D}_{w_0} Y(s), s \leq 0)$. In particular, the random variable $V_1(0) = \inf_{t>0} \beta_1^\vee(X(t))$ is measurable with respect to the σ-field generated by $(\mathscr{D}_{w_0} Y(s), s \leq 0)$. Now, for each $\alpha \in S$, there is a reduced decomposition of w_0 with $\beta_1 = \alpha$, so we see that the random variables $\inf_{t\geq 0} \alpha^\vee(X(t))$, α simple root, are all measurable with respect to the σ-field generated by $(\mathscr{D}_{w_0} Y(s), s \leq 0)$ and therefore independent of $(\mathscr{D}_{w_0} Y(s), s \geq 0)$, as required. Thus Lemma 5.4 is proven. □

THEOREM 5.6
Let X be a Brownian motion in V. Then $\mathscr{P}_{w_0} X$ is a Brownian motion in $\overline{C}$.

Proof
Let $x, \xi \in C$, and let X be a Brownian motion with drift ξ. The event "X remains in the cone $C - x$ for all times" can be expressed in terms of the variables $\inf_{t\geq 0} \alpha^\vee(X(t))$, α simple root; therefore, by Lemma 5.4, it is independent of $(\mathscr{D}_{w_0}X(t), t \geq 0)$. Thus if R has the same distribution as that of X conditioned on this event, then $\mathscr{D}_{w_0}R$ has the same distribution as X. Now we can let $x, \xi \to 0$ so that X is a Brownian motion with no drift and R is a Brownian motion in $\overline{C}$; by continuity, $\mathscr{D}_{w_0}R$ has the same distribution as X. Now, by Lemma 5.3, $\mathscr{Q}_{w_0}\mathscr{D}_{w_0}R = R$ a.s. It follows that $\mathscr{Q}_{w_0}X$ and hence $\mathscr{P}_{w_0}X$ are Brownian motions in $\overline{C}$, as required. □

5.5. Random walks and Markov chains on the weight lattice
We now present the second proof of the Brownian motion property. We assume that W is the Weyl group of the semisimple Lie algebra $\mathfrak{g}$, as in Sections 3.1 and 5.1, and that $V = \mathfrak{a}^*$. As in Section 5.1, let T be a maximal torus of the compact group K, the simply connected compact group with Lie algebra $\mathfrak{g}_{\mathbb{R}}$, a compact form of $\mathfrak{g}$. Let $\omega \in P_+$ be a nonzero dominant weight, and let χ_ω be the character of the associated highest weight module. As a function on T, this is the Fourier transform of the positive measure R_ω on P, which puts a weight m_μ^ω on a weight μ, where m_μ^ω is the multiplicity of μ in the module with highest weight ω. In other words,

$$\chi_\omega = \sum_{\mu \in P_+} m_\mu^\omega e(\mu),$$

where $e(\mu)(\theta) = e^{2i\pi\langle\mu,\theta\rangle}$ is the character on T. We can divide this measure R_ω by $\dim\omega$ to get a probability measure

$$\nu_\omega = \frac{1}{\dim\omega} R_\omega.$$

Consider the random walk $X_n, n \geq 0$, on the weight lattice whose increments are distributed according to this probability measure, started at zero. Thus the transition probabilities of this random walk are given by

$$p_\omega(\mu, \lambda) = \frac{m_{\lambda-\mu}^\omega}{\dim\omega}.$$

Donsker's theorem and invariance of m^ω under the Weyl group imply the following theorem.

THEOREM 5.7
The stochastic process $X_{[Nt]}/\sqrt{N}$ converges, as $N \to \infty$, to a Brownian motion on $\mathfrak{a}^*$ with correlation invariant under W.

Let us define a probability transition function q_ω on P_+ by the formula

$$\frac{\chi_\mu}{\dim \mu}\frac{\chi_\omega}{\dim \omega} = \sum_{\lambda \in P_+} q_\omega(\mu, \lambda)\frac{\chi_\lambda}{\dim \lambda}.$$

Thus $q_\omega(\mu, \lambda)$ is equal to $M^\lambda_{\omega,\mu} \dim \lambda / \dim \omega \dim \mu$, where $M^\lambda_{\omega,\mu}$ is the multiplicity of the module with highest weight λ in the decomposition of the tensor product of the modules with highest weights ω and μ (see, e.g., [11], [5]).

LEMMA 5.8
One has

$$q_\omega(\mu, \lambda) = \frac{\dim \lambda}{\dim \mu} \sum_{w \in W} \varepsilon(w) p_\omega\big(\mu + \rho, w(\lambda + \rho)\big).$$

Proof
Let dk be the normalized Haar measure on K. By the orthogonality relations for characters, one has

$$M^\lambda_{\omega,\mu} = \int_K \overline{\chi_\lambda}(k)\chi_\mu(k)\chi_\omega(k)\, dk.$$

Therefore

$$q_\omega(\mu, \lambda) = \frac{M^\lambda_{\omega,\mu} \dim \lambda}{\dim \omega \dim \mu} = \frac{\dim \lambda}{\dim \mu \dim \omega} \int_K \overline{\chi_\lambda}(k)\chi_\mu(k)\chi_\omega(k)\, dk.$$

Now we can use the Weyl integration formula as well as the Weyl character formula to rewrite the formula as an integral over T, the maximal torus of K. Thus

$$\begin{aligned} q_\omega(\mu, \lambda) = \frac{|W| \dim \lambda}{\dim \mu \dim \omega} \int_T \sum_{w_1, w_2 \in W} &\varepsilon(w_1)\varepsilon(w_2)\overline{e\big(w_1(\lambda + \rho)\big)}(\theta) e \\ &\times \big(w_2(\mu + \rho)\big)(\theta)\chi_\omega(\theta)\, d\theta, \end{aligned}$$

where $e(\gamma)(\theta) = e^{2i\pi\langle \gamma, \theta \rangle}$ and ρ is half the sum of positive weights. Now using the invariance of χ_ω under the Weyl group, we can rewrite this as

$$\begin{aligned} q_\omega(\mu, \lambda) &= \frac{\dim \lambda}{\dim \mu \dim \omega} \int_T \sum_{w \in W} \varepsilon(w)\overline{e(\lambda + \rho)}(\theta) e\big(w(\mu + \rho)\big)(\theta)\chi_\omega(\theta)\, d\theta \\ &= \frac{\dim \lambda}{\dim \mu} \sum_{w \in W} p_\omega\big(\mu + \rho, w(\lambda + \rho)\big). \qquad \square \end{aligned}$$

From (5.2), Theorem 5.7, Lemma 5.8, and standard arguments, we deduce the following proposition.

PROPOSITION 5.9
Let Y be a Markov chain on P_+ started at zero, with transition probabilities $q_\omega(\mu, \lambda)$; then $Y([Nt])/\sqrt{N}$ converges in distribution, as $N \to \infty$ to a Brownian motion in the Weyl chamber $\overline{C}$.

5.6. Pitman operators and the Markov chain on the weight lattice
We choose a nonzero dominant weight ω and a dominant path π^ω defined on $[0, 1]$ with $\pi^\omega(1) = \omega$. Let $B\pi^\omega$ be the set of paths in the Littelmann module generated by π^ω. We now construct a stochastic process with values in P. Choose independent random paths $\eta_n \in B\pi^\omega, n = 1, 2, \dots$, each with uniform distribution on $B\pi^\omega$, and define the stochastic process Z as the random path obtained by the usual concatenations $\eta_1 * \eta_2 * \cdots$ of the $\eta_i, i = 1, 2, \dots$. In other words, one has $Z(t) = \eta_1(1) + \eta_2(1) + \cdots + \eta_{n-1}(1) + \eta_n(t-n)$ if $t \in [n, n+1]$. Beware that this concatenation does not coincide with Littelmann's definition, recalled in Section 4.4, since we do not rescale the time. Littelmann's theory then implies that $\eta_n(1)$ is a random weight in P with distribution ν_ω and that $Z(n), n = 0, 1, \dots,$ is the random walk in $\mathfrak{a}^*$ with this distribution of increments.

THEOREM 5.10
The stochastic process $\mathscr{P}_{w_0}Z(n), n = 0, 1, \dots,$ is a Markov chain on P_+ with probability transitions q_ω.

Proof
First note that the set of paths of the form $\eta_1 * \eta_2 * \cdots * \eta_n$, where $\eta_i \in B\pi^\omega$, is stable under Littelmann operators by [22]; therefore, by (2.3), it is also stable under Pitman transformations. Consider a dominant path of the form $\gamma_1 * \gamma_2 * \cdots * \gamma_n$ with all $\gamma_i \in B\pi^\omega$. We compute the conditional probability distribution of $\mathscr{P}_{w_0}Z(n+1)$ knowing that $\mathscr{P}_{w_0}Z(t) = \gamma_1 * \gamma_2 * \cdots * \gamma_n(t)$ for $t \leq n$. Let $\mu = \gamma_1 * \gamma_2 * \cdots * \gamma_n(1)$. By Corollary 2.10, the set of all paths of the form $\eta_1 * \eta_2 * \cdots * \eta_n$ such that $\mathscr{P}_{w_0}(\eta_1 * \eta_2 * \cdots * \eta_n) = \gamma_1 * \gamma_2 * \cdots * \gamma_n$ coincides with the Littelmann module $B(\gamma_1 * \gamma_2 * \cdots * \gamma_n)$. Now consider a path $\eta_{n+1} \in B\pi$ and the concatenation $\eta_1 * \eta_2 * \cdots * \eta_n * \eta_{n+1}$; then $\mathscr{P}_{w_0}(\eta_1 * \eta_2 * \cdots * \eta_n * \eta_{n+1})$ is the dominant path in the Littelmann module generated by $\eta_1 * \eta_2 * \cdots * \eta_n * \eta_{n+1}$. By Littelmann's version of the Littlewood-Richardson rule (see [22, Sec. 10]), the number of pairs of paths $(\eta_1 * \eta_2 * \cdots * \eta_n, \eta_{n+1})$ such that $\mathscr{P}_{w_0}(\eta_1 * \eta_2 * \cdots * \eta_n) = \gamma_1 * \gamma_2 * \cdots * \gamma_n$ and $\mathscr{P}_{w_0}(\eta_1 * \eta_2 * \cdots * \eta_n * \eta_{n+1})(1) = \lambda$ is equal to the dimension of the isotypic component of type λ in the module which is the tensor product of the highest weight modules μ and ω; in particular, this depends only on μ and is equal to $M^\lambda_{\omega,\mu} \dim \lambda$. Since the total number of pairs $(\eta_1 * \eta_2 * \cdots * \eta_n, \eta_{n+1})$ with $\mathscr{P}_{w_0}(\eta_1 * \eta_2 * \cdots * \eta_n) = \gamma_1 * \gamma_2 * \cdots * \gamma_n$ is $\dim \mu \dim \omega$, we see that the

conditional probability we seek is $M^{\lambda}_{\omega,\mu}\dim\lambda/\dim\omega\dim\mu = q_{\omega}(\mu,\lambda)$. This proves the claim. $\square$

5.7. Second proof of the representation theorem for Weyl groups
Putting together Proposition 5.9 and Theorem 5.10, we get another proof of Theorem 5.6. Indeed, by Donsker's theorem, the process $Z([Nt])/\sqrt{N}$ gives as limit the Brownian motion in $\mathfrak{a}^*$. By Theorem 5.10, the process $\mathscr{P}_{w_0}Z(n)$, $n \geq 0$, is distributed as the Markov process of Proposition 5.9. Applying the scaling of Proposition 5.9 to the stochastic process $\mathscr{P}_{w_0}Z(t)$, $t \geq 0$, yields for limit process the Brownian motion on the Weyl chamber. Since $\mathscr{P}_{w_0}$ is a continuous map that commutes with scaling, we get the proof of Theorem 5.6 when W is the Weyl group of a complex semisimple Lie algebra. $\square$

5.8. A remark on the Duistermaat-Heckman measure
The distribution of the path $t \in [0,n] \mapsto Z(t)$ is uniform on the set

$$B(\pi^{\omega})^{*n} = \{\eta_1 * \eta_2 * \cdots * \eta_n; \eta_i \in B\pi^{\omega}\}.$$

Therefore, for any path $\eta \in B(\pi^{\omega})^{*n}$, the distribution of $(Z(s))_{0\leq s\leq n}$, conditionally on $\mathscr{P}_{w_0}Z(s) = \eta(s), 0 \leq s \leq n$, is uniform on the set $\gamma \in B(\pi^{\omega})^{*n}; \mathscr{P}_{w_0}\gamma = \eta$. It thus follows from the Littelmann theory (see [22]) that the conditional distribution of the terminal value Z_n is the probability measure ν_{η}. It has been proved by Heckman [17] (see also [16], [10]) that if $\gamma_{\varepsilon} \to \infty$ in $\mathfrak{a}^*_+$ and $\varepsilon\gamma_{\varepsilon} \to v$, then $D_{\varepsilon}\nu_{\gamma_{\varepsilon}}$ converges to the so-called Duistermaat-Heckman (DH) measure associated to v, that is, the projection of the normalized measure on the coadjoint orbit of K through v, by the orthogonal projection on $\mathfrak{a}^*$. This follows from Kirillov's character formula for K. From Section 5.7, we deduce that if X is the Brownian motion on $\mathfrak{a}^*$, then the law of $X(T)$, conditionally on $\mathscr{P}_{w_0}X = \gamma$ on $[0,T]$, is the DH measure associated with $\gamma(T)$.

A. Appendix. Proof of Proposition 2.2(iv)

Let η be a path. Defining $\pi = \mathscr{P}_{\alpha}\eta$, $x = -\inf_{T\geq t\geq 0}\alpha^{\vee}(\eta(t))$, and $t_0 = \sup\{t|\alpha^{\vee}(\eta(t)) = -x\}$, we check that equation (2.2) is valid.

If $t \geq t_0$, then one has $\inf_{0\leq s\leq t}\alpha^{\vee}(\eta(s)) = -x$. Therefore

$$\begin{aligned}\alpha^{\vee}\big(\pi(t)\big) &= \alpha^{\vee}\big(\eta(t)\big) + 2x\\ &= x + \big(\alpha^{\vee}(\eta(t)) + x\big)\\ &\geq x\end{aligned}$$

for all $t \geq t_0$. It follows that $\inf\big(x, \inf_{T\geq s\geq t}\alpha^{\vee}(\pi(s))\big) = x$ for $t \geq t_0$. Formula (2.2) follows for $t \geq t_0$.

If $t < t_0$, let $u = \inf\{s \geq t | \alpha^\vee(\eta(s)) = \inf_{0\leq v\leq t} \alpha^\vee(\eta(v))\}$. Then $t \leq u \leq t_0$. One has

$$\begin{aligned}\alpha^\vee\big(\pi(u)\big) &= \alpha^\vee\big(\eta(u)\big) - 2\inf_{0\leq v\leq u} \alpha^\vee\big(\eta(v)\big)\\ &= -\alpha^\vee\big(\eta(u)\big),\end{aligned}$$

which implies that $\inf_{T\geq v\geq t} \alpha^\vee(\pi(v)) \leq -\inf_{0\leq v\leq t} \alpha^\vee(\eta(v)) \leq x$. On the other hand, for $v \geq t$, one has

$$\begin{aligned}\alpha^\vee\big(\pi(v)\big) &= \alpha^\vee\big(\eta(v)\big) - 2\inf_{0\leq s\leq v} \alpha^\vee\big(\eta(s)\big)\\ &\geq \big(\alpha^\vee(\eta(v)) - \inf_{0\leq s\leq v} \alpha^\vee(\eta(s))\big) - \inf_{0\leq s\leq t} \alpha^\vee\big(\eta(s)\big)\\ &\geq -\inf_{0\leq s\leq t} \alpha^\vee\big(\eta(s)\big).\end{aligned}$$

Therefore $\inf_{T\geq v\geq t} \alpha^\vee(\pi(v)) = -\inf_{0\leq s\leq t} \alpha^\vee(\eta(s))$, and formula (2.2) for $t < t_0$ follows. The existence and uniqueness in Proposition 2.2 follows. □

Acknowledgments. We thank P. Littelmann for a useful conversation at an early stage of this work and P. Diaconis and S. Evans for their encouragement and support. We also thank the referees for useful comments.

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