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Continuous crystal and Duistermaat–Heckman measure for Coxeter groups

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Abstract

We introduce a notion of continuous crystal analogous, for general Coxeter groups, to the combinatorial crystals introduced by Kashiwara in representation theory of Lie algebras. We explore their main properties in the case of finite Coxeter groups, where we use a generalization of the Littelmann path model to show the existence of the crystals. We introduce a remarkable measure, analogous to the Duistermaat–Heckman measure, which we interpret in terms of Brownian motion. We also show that the Littelmann path operators can be derived from simple considerations on Sturm–Liouville equations.

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Contents

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1. Introduction

1.1. The aim of this paper is to introduce a notion of continuous crystals for Coxeter groups, which are not necessarily Weyl groups. Crystals are combinatorial objects, which have been associated by Kashiwara to Kac–Moody algebras, in order to provide a combinatorial model for the representation theory of these algebras, see, e.g., [15,17,18,21] for an introduction to this theory. The crystal graphs defined by Kashiwara turn out to be equivalent to certain other graphs, constructed independently by Littelmann, using his path model. The approach of Kashiwara to the crystals is through representations of quantum groups and their "crystallization," which is the process of letting the parameter q in the quantum group go to zero. This requires representation theory and therefore does not make sense for realizations of arbitrary Coxeter groups. On the other hand, as it was realized in a previous paper [3], Littelmann's model can be adapted to fit with non-crystallographic Coxeter groups, but the price to pay is that, since there is no lattice invariant under the action of the group, one can only define a continuous version of the path model, namely of the Littelmann path operators (see however the recent preprint [19], which has appeared when this paper was under revision). In this continuous model, instead of the Littelmann path operators e_i, f_i we have continuous semigroups e_i^t, f_i^t indexed by nonnegative real numbers $t \geqslant 0$. In the crystallographic case it is possible to think of these continuous crystals as "semi-classical limits" of the combinatorial crystals, in much the same way as the coadjoint orbits arise as semi-classical limits of the representations of a compact semi-simple Lie group. These continuous path operators, and the closely related Pitman transforms, were used in [3] to investigate symmetry properties of Brownian motion in a space where a finite Coxeter group acts, with applications in particular to the motion of eigenvalues of matrix-valued Brownian motions. In this paper, which is a sequel to [3], but can for the most part be read independently, we define continuous crystals and start investigating their main properties. As for now the theory works well for finite Coxeter groups, but there are still several difficulties to extend it to infinite groups. This theory allows us to define objects which are analogues to simplified versions of the Schubert varieties (or Demazure–Littelmann modules) associated with semi-simple Lie groups. We hope these objects might help in certain questions concerning Coxeter groups, such as, for example, the Kazhdan–Lusztig polynomials.

1.2. This paper is organized as follows. The next section contains the main definition, that of a continuous crystal associated with a realization of a Coxeter group. We establish the main properties of these objects, following closely the exposition of Joseph in [18]. It would have been possible to just refer to [18] for the most part of this section, however, for the convenience of the reader, and also for convincing ourselves that everything from the crystallographic situation goes smoothly to the continuous context, we have preferred to write everything down. The main body of the proof is relegated to Appendix A in order to ease the reading of the paper. The main result of this section is Theorem 2.6, a uniqueness result for continuous crystals, analogous to the one in [18]. In Section 3 we introduce the path operators and establish their most important properties. Our approach to the path model is different from that in Littelmann [23], in that we base our exposition on the Pitman transforms, which are defined from scratch. These transforms satisfy braid relations, which where proved in [3], and which play a prominent role. Using these operators, the set of continuous paths is endowed with a crystal structure and the continuous analogues of the Littelmann modules are introduced as "connected components" of this crystal (see the discussion following Proposition 3.9, Definition 3.10 and Theorem 3.11). Our definition makes sense for arbitrary Coxeter groups, but we are able to prove significant properties of these only in the case of finite Coxeter groups. It remains an interesting and challenging problem to extend these properties to the general case. Continuous Littelmann modules can be parametrized in several ways by polytopes, corresponding to different reduced decompositions of an element in the Coxeter group. In the case of Weyl groups, these are the Berenstein–Zelevinsky polytopes (see [2]) which contain the Kashiwara coordinates on the crystals. In Section 4 we state some properties of these parametrizations. In Theorem 3.12 we prove that two such parametrizations are related by a piecewise linear transformation, and in Theorem 4.5 we show that the polytopes can be obtained by the intersection of a cone depending only on the element of the Coxeter group, and a set of inequalities which depend on the dominant path. Furthermore, we provide explicit equations for the cone in the dihedral case (in Proposition 4.7). In Theorem 4.9 we prove that the crystal associated with a Littelmann module depends only on the end point of the dominant path, then in Theorem 4.14 we obtain the existence and uniqueness of a family of highest weight normal continuous crystals. We show that the Coxeter group acts on each Littelmann module (Theorem 4.16). We introduce the Schützenberger involution in Section 4.10 and use it to give a direct combinatorial proof of the commutativity of the tensor product of continuous crystals (Theorem 4.20). We think that even in the crystallographic case our treatment sheds some light on these topics. In Section 5, we introduce an analogue of the Duistermaat–Heckman measure, motivated by a result of Alexeev and Brion [1]. We prove several interesting properties of this measure, in particular, in Theorem 5.5, an analogue of the Harish-Chandra formula. The Laplace transform appearing in this formula is a generalized Bessel function. It is shown in Theorem 5.16 to satisfy a product formula, giving a positive answer to a question of Rösler. The Duistermaat–Heckman measure is intimately linked with Brownian motion, and in Corollary 5.3 we give a Brownian proof of the fact that the crystal defined by the path model depends only on the final position of the path. The final section is of a quite different nature, and somewhat independent of the rest of the paper. The Littelmann path operators have been introduced as a generalization, for arbitrary root systems, of combinatorial operations on Young tableaux. Here we show how, using some simple considerations on Sturm–Liouville equations, the Littelmann path operators appear naturally. In particular this gives a concrete geometric basis to the theory of geometric lifting which has been introduced by Berenstein and Zelevinsky in [2] in a purely formal way.

2. Continuous crystal

This section is devoted to introducing the main definition and first properties of continuous crystals.

2.1. Basic definition

We use the standard references [4,16] on Coxeter groups and their realizations. A Coxeter system (W, S) is a group W generated by a finite set of involutions S such that, if $m(s, s')$ is the order of ss' then the relations

$$(ss')^{m(s,s')} = 1$$

for $m(s, s')$ finite, give a presentation of W.

A realization of (W, S) is given by a real vector space V with dual $V^\vee$, an action of W on V, and a subset $\{(\alpha_s, \alpha_s^\vee),\ s \in S\}$ of $V \times V^\vee$ such that each $s \in S$ acts on V by the reflection given by

$$s(x) = x - \alpha_s^\vee(x)\alpha_s, \quad x \in V,$$

so $\alpha_s^\vee(\alpha_s) = 2$. One calls α_s the simple root associated with $s \in S$ and $\alpha_s^\vee$ its coroot.

We consider a realization of a Coxeter system (W, S) in a real vector space V, and the associated simple roots $\Sigma = \{\alpha_s,\ s \in S\}$ in V and coroots $\{\alpha_s^\vee,\ s \in S\}$ in $V^\vee$. The closed Weyl chamber is the convex cone

$$\overline{C} = \{v \in V;\ \alpha_s^\vee(v) \geqslant 0,\ \text{for all } \alpha \in S\}$$

thus the simple roots are positive on $\overline{C}$. There is an order relation on V induced by this cone, namely $\lambda \leqslant \mu$ if and only if $\mu - \lambda \in \overline{C}$.

We adapt the definition of crystals due to Kashiwara (see, e.g., Kashiwara [20,21], Joseph [17]) to a continuous setting.

Definition 2.1. A continuous crystal is a set B equipped with maps

$$wt : B \to V,$$
$$\varepsilon_\alpha, \varphi_\alpha : B \to \mathbb{R} \cup \{-\infty\}, \quad \alpha \in \Sigma,$$
$$e_\alpha^r : B \cup \{\mathbf{0}\} \to B \cup \{\mathbf{0}\}, \quad \alpha \in \Sigma,\ r \in \mathbb{R},$$

where $\mathbf{0}$ is a ghost element, such that the following properties hold, for all $\alpha \in \Sigma$, and $b \in B$:

(C1) $\varphi_\alpha(b) = \varepsilon_\alpha(b) + \alpha^\vee(wt(b))$.

(C2) If $e_\alpha^r(b) \neq \mathbf{0}$ then

$$\varepsilon_\alpha(e_\alpha^r b) = \varepsilon_\alpha(b) - r,$$
$$\varphi_\alpha(e_\alpha^r b) = \varphi_\alpha(b) + r,$$
$$wt(e_\alpha^r b) = wt(b) + r\alpha.$$

(C3) For all $r \in \mathbb{R}$, $b \in B$ one has $e_\alpha^r(\mathbf{0}) = \mathbf{0}$, $e_\alpha^0(b) = b$. If $e_\alpha^r(b) \neq \mathbf{0}$ then, for all $s \in \mathbb{R}$,

$$e_\alpha^{s+r}(b) = e_\alpha^s(e_\alpha^r(b)).$$

(C4) If $\varphi_\alpha(b) = -\infty$ then $e^r_\alpha(b) = \mathbf{0}$, for all $r \in \mathbb{R}$, $r \neq 0$.

The point is that, in this definition, r takes any real value, and not only discrete ones. Sometimes we write, for $r \geqslant 0$,

$$f^r_\alpha = e^{-r}_\alpha.$$

Example 2.2 *(The crystal B_α).* For each $\alpha \in \Sigma$, we define the crystal B_α as the set $\{b_\alpha(t)$, t is a nonpositive real number$\}$, with the maps given by

$$wt\big(b_\alpha(t)\big) = t\alpha, \qquad \varepsilon_\alpha\big(b_\alpha(t)\big) = -t, \qquad \varphi_\alpha\big(b_\alpha(t)\big) = t,$$
$$e^r_\alpha\big(b_\alpha(t)\big) = b_\alpha(t+r) \quad \text{if } r \leqslant -t \quad \text{and} \quad e^r_\alpha\big(b_\alpha(t)\big) = \mathbf{0}, \quad \text{otherwise},$$

and, if $\alpha' \neq \alpha$, $\varepsilon_{\alpha'}(b_\alpha(t)) = -\infty$, $\varphi_{\alpha'}(b_\alpha(t)) = -\infty$, $e^r_{\alpha'}(b_\alpha(t)) = \mathbf{0}$, when $r \neq 0$.

2.2. Morphisms

Definition 2.3. Let B_1 and B_2 be continuous crystals.

1. A morphism of crystals $\psi : B_1 \to B_2$ is a map $\psi : B_1 \cup \{\mathbf{0}\} \to B_2 \cup \{\mathbf{0}\}$ such that $\psi(\mathbf{0}) = \mathbf{0}$ and for all $\alpha \in \Sigma$ and $b \in B_1$,

$$wt\big(\psi(b)\big) = wt(b), \qquad \varepsilon_\alpha\big(\psi(b)\big) = \varepsilon_\alpha(b), \qquad \varphi_\alpha\big(\psi(b)\big) = \varphi_\alpha(b)$$

   and $e^r_\alpha(\psi(b)) = \psi(e^r_\alpha(b))$ when $e^r_\alpha(b) \in B_1$.
2. A strict morphism is a morphism $\psi : B_1 \to B_2$ such that $e^r_\alpha(\psi(b)) = \psi(e^r_\alpha(b))$ for all $b \in B_1$.
3. A crystal embedding is an injective strict morphism.

The morphism ψ is called a *crystal isomorphism* if there exists a crystal morphism $\phi : B_2 \to B_1$ such that $\phi \circ \psi = id_{B_1 \cup \{\mathbf{0}\}}$, and $\psi \circ \phi = id_{B_2 \cup \{\mathbf{0}\}}$. It is then an embedding.

2.3. Tensor product

Consider two continuous crystals B_1 and B_2 associated with (W, S, Σ). We define the tensor product $B_1 \otimes B_2$ as the continuous crystal with set $B = B_1 \times B_2$, whose elements are denoted $b_1 \otimes b_2$, for $b_1 \in B_1$, $b_2 \in B_2$. Let $\sigma = \varphi_\alpha(b_1) - \varepsilon_\alpha(b_2)$ where $(-\infty) - (-\infty) = 0$, let $\sigma^+ = \max(0, \sigma)$ and $\sigma^- = \max(0, -\sigma)$, then the maps defining the tensor product are given by the following formulas:

$$wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2),$$
$$\varepsilon_\alpha(b_1 \otimes b_2) = \varepsilon_\alpha(b_1) + \sigma^-,$$
$$\phi_\alpha(b_1 \otimes b_2) = \phi_\alpha(b_2) + \sigma^+,$$
$$e^r_\alpha(b_1 \otimes b_2) = e_\alpha^{\max(r,-\sigma)-\sigma^-} b_1 \otimes e_\alpha^{\min(r,-\sigma)+\sigma^+} b_2.$$

Here $b_1 \otimes \mathbf{0}$ and $\mathbf{0} \otimes b_2$ are understood to be $\mathbf{0}$. Notice that when $\sigma \geqslant 0$, one has $\varepsilon_\alpha(b_1 \otimes b_2) = \varepsilon_\alpha(b_1)$ and

$$e_\alpha^r(b_1 \otimes b_2) = e_\alpha^r b_1 \otimes b_2, \quad \text{for all } r \in [-\sigma, +\infty[. \tag{2.1}$$

As in the discrete case, one can check that the tensor product is associative (but not commutative) so we can define without ambiguity the tensor product of several crystals.

2.4. *Highest weight crystal*

A crystal B is called upper normal when, for all $b \in B$,

$$\varepsilon_\alpha(b) = \max\{r \geqslant 0;\ e_\alpha^r(b) \neq \mathbf{0}\}$$

and is called lower normal if

$$\varphi_\alpha(b) = \max\{r \geqslant 0;\ e_\alpha^{-r}(b) \neq \mathbf{0}\}.$$

We call it normal (this is sometimes called seminormal by Kashiwara) when it is lower and upper normal. Notice that this implies that $\varepsilon_\alpha(b) \geqslant 0$ and $\varphi_\alpha(b) \geqslant 0$.

We introduce the semigroup $\mathcal{F}$ generated by the $\{f_\alpha^r, \alpha \text{ simple root}, r \geqslant 0\}$:

$$\mathcal{F} = \{f_{\alpha_1}^{r_1} \cdots f_{\alpha_k}^{r_k},\ k \in \mathbb{N}^*,\ r_1, \dots, r_k \geqslant 0,\ \alpha_1, \dots, \alpha_k \in \Sigma\},$$

and, if b is an element of a continuous crystal B, the subset $\mathcal{F}(b) = \{f(b),\ f \in \mathcal{F}\}$ of B.

Definition 2.4. Let $\lambda \in V$, a continuous crystal $B(\lambda)$ is said to be of highest weight λ if there exists $b_\lambda \in B(\lambda)$ such that $wt(b_\lambda) = \lambda$, $e_\alpha^r(b_\lambda) = \mathbf{0}$, for all $r > 0$ and $\alpha \in \Sigma$ and such that $B(\lambda) = \mathcal{F}(b_\lambda)$.

For a continuous crystal with highest weight λ, such an element b_λ is unique, and called the primitive element of $B(\lambda)$. If the crystal is normal then λ must be in the Weyl chamber $\overline{C}$. The vector λ is a highest weight in the sense that, for all $b \in B(\lambda)$, $wt(b) \leqslant \lambda$.

2.5. *Uniqueness*

Following Joseph [17,18] we introduce the following definition.

Definition 2.5. Let $(B(\lambda),\ \lambda \in \overline{C})$, be a family of highest weight continuous crystals. The family is closed if, for each $\lambda, \mu \in \overline{C}$, the subset $\mathcal{F}(b_\lambda \otimes b_\mu)$ of $B(\lambda) \otimes B(\mu)$ is a crystal isomorphic to $B(\lambda + \mu)$.

Joseph [17, 6.4.21], has shown in the Weyl group case, for discrete crystals, that a closed family of highest weight normal crystals is unique. The analogue holds in our situation.

Theorem 2.6. *For a realization of a Coxeter system (W, S), if a closed family $B(\lambda)$, $\lambda \in \overline{C}$, of highest weight continuous normal crystals exists, then it is unique.*

The proof of the theorem, which follows closely Joseph [18], is in Appendix A.1.

3. Pitman transforms and Littelmann path operators for Coxeter groups

In this section we recall definition and properties of Pitman transforms, introduced in our previous paper [3]. We deduce from these properties the existence of Littelmann operators, then we define continuous Littelmann modules, prove that they are continuous crystals, and make a first study of their parametrization.

3.1. The Pitman transform

Let V be a real vector space, with dual space $V^\vee$. Let $\alpha \in V$ and $\alpha^\vee \in V^\vee$ be such that $\alpha^\vee(\alpha) = 2$. The reflection $s_\alpha : V \to V$ associated to $(\alpha, \alpha^\vee)$ is the linear map defined, for $x \in V$, by

$$s_\alpha(x) = x - \alpha^\vee(x)\alpha.$$

For $T > 0$, let $C_T^0(V)$ be the set of continuous path $\eta : [0, T] \to V$ such that $\eta(0) = 0$, with the topology of uniform convergence. We have introduced and studied in [3] the following path transformation, similar to the one defined by Pitman in [30].

Definition 3.1. The Pitman transform $\mathcal{P}_\alpha$ associated with $(\alpha, \alpha^\vee)$ is defined on $C_T^0(V)$ by the formula:

$$\mathcal{P}_\alpha \eta(t) = \eta(t) - \inf_{t \geqslant s \geqslant 0} \alpha^\vee\big(\eta(s)\big)\alpha, \quad T \geqslant t \geqslant 0.$$

A path $\eta \in C_T^0(V)$ is called α-dominant when $\alpha^\vee(\eta(t)) \geqslant 0$ for all $t \in [0, T]$. The following properties of the Pitman transform are easily established.

Proposition 3.2.

(i) *The transformation $\mathcal{P}_\alpha : C_T^0(V) \to C_T^0(V)$ is continuous.*
(ii) *For all $\eta \in C_T^0(V)$, the path $\mathcal{P}_\alpha \eta$ is α-dominant and $\mathcal{P}_\alpha \eta = \eta$ if and only if η is α-dominant.*
(iii) *The transformation $\mathcal{P}_\alpha$ is an idempotent, i.e. $\mathcal{P}_\alpha \mathcal{P}_\alpha \eta = \mathcal{P}_\alpha \eta$ for all $\eta \in C_T^0(V)$.*
(iv) *Let $\pi \in C_T^0(V)$ be α-dominant, and let $x \in [0, \alpha^\vee(\pi(T))]$, then there exists a unique path η in $C_T^0(V)$ such that $\mathcal{P}_\alpha \eta = \pi$ and $\eta(T) = \pi(T) - x\alpha$. Moreover for $0 \leqslant t \leqslant T$,*

$$\eta(t) = \pi(t) - \min\Big[x, \inf_{T \geqslant s \geqslant t} \alpha^\vee\big(\pi(s)\big)\Big]\alpha.$$

3.2. Littelmann path operators

Let $V, V^\vee, \alpha, \alpha^\vee$ be as above. Using Proposition 3.2, as in [3], we can define generalized Littelmann path operators (see [23]).

Definition 3.3. Let $\eta \in C_T^0(V)$, and $x \in \mathbb{R}$, then we define $\mathcal{E}_\alpha^x \eta$ as the unique path such that

$$\mathcal{P}_\alpha \mathcal{E}_\alpha^x \eta = \mathcal{P}_\alpha \eta \quad \text{and} \quad \mathcal{E}_\alpha^x \eta(T) = \eta(T) + x\alpha$$

if $-\alpha^\vee(\eta(T)) + \inf_{0\leqslant t\leqslant T}\alpha^\vee(\eta(t)) \leqslant x \leqslant -\inf_{0\leqslant t\leqslant T}\alpha^\vee(\eta(t))$ and $\mathcal{E}_\alpha^x\eta = \mathbf{0}$ otherwise. The following formula holds:

$$\mathcal{E}_\alpha^x\eta(t) = \eta(t) - \min\Big(-x, \inf_{t\leqslant s\leqslant T}\alpha^\vee\big(\eta(s)\big) - \inf_{0\leqslant s\leqslant T}\alpha^\vee\big(\eta(s)\big)\Big)\alpha$$

if $-\alpha^\vee(T) + \inf_{0\leqslant t\leqslant T}\alpha^\vee(\eta(t)) \leqslant x \leqslant 0$, and

$$\mathcal{E}_\alpha^x\eta(t) = \eta(t) - \min\Big(0, -x - \inf_{0\leqslant s\leqslant T}\alpha^\vee\big(\eta(s)\big) + \inf_{0\leqslant s\leqslant t}\alpha^\vee\big(\eta(s)\big)\Big)\alpha$$

if $0 \leqslant x \leqslant -\inf_{0\leqslant t\leqslant T}\alpha^\vee(\eta(t))$.

Here, as in the definition of crystals, $\mathbf{0}$ is a ghost element. The following result is immediate from the definition of the Littelmann operators.

Proposition 3.4. *$\mathcal{E}_\alpha^0\eta = \eta$ and $\mathcal{E}_\alpha^x\mathcal{E}_\alpha^y\eta = \mathcal{E}_\alpha^{x+y}\eta$ as long as $\mathcal{E}_\alpha^y\eta \neq \mathbf{0}$.*

We shall also use the notation $\mathcal{F}_\alpha^x = \mathcal{E}_\alpha^{-x}$ for $x \geqslant 0$, and denote by $\mathcal{H}_\alpha^x$ the restriction of the operator $\mathcal{F}_\alpha^x$ to α-dominant paths. Let π be an α-dominant path in $C_T^0(V)$ and $0 \leqslant x \leqslant \alpha^\vee(T)$, then $\mathcal{H}_\alpha^x\pi$ is the unique path in $C_T^0(V)$ such that

$$\mathcal{P}_\alpha\mathcal{H}_\alpha^x\pi = \pi$$

and

$$\mathcal{H}_\alpha^x\pi(T) = \pi(T) - x\alpha.$$

Observe that in this equality

$$x = -\inf_{0\leqslant t\leqslant T}\alpha^\vee\big(\mathcal{H}_\alpha^x\pi(t)\big).$$

3.3. Product of Pitman transforms

Let $\alpha, \beta \in V$ and $\alpha^\vee, \beta^\vee \in V^\vee$ be such that $\alpha^\vee(\beta) < 0$ and $\beta^\vee(\alpha) < 0$. Replacing if necessary $(\alpha, \alpha^\vee, \beta, \beta^\vee)$ by $(t\alpha, \alpha^\vee/t, \beta/t, t\beta^\vee)$, which does not change $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$, we will assume that $\alpha^\vee(\beta) = \beta^\vee(\alpha)$. We use the notations $\rho = -\frac{1}{2}\alpha^\vee(\beta) = -\frac{1}{2}\beta^\vee(\alpha)$. The following result is proved in [3].

Theorem 3.5. *Let n be a positive integer, then if $\rho \geqslant \cos\frac{\pi}{n}$,*

$$\underbrace{(\mathcal{P}_\alpha\mathcal{P}_\beta\mathcal{P}_\alpha\dots)}_{n\ terms}\pi(t) = \pi(t) - \inf_{t\geqslant s_0\geqslant s_1\geqslant\cdots\geqslant s_{n-1}\geqslant 0}\left(\sum_{i=0}^{n-1}T_i(\rho)Z^{(i)}(s_i)\right)\alpha$$
$$- \inf_{t\geqslant s_0\geqslant s_1\geqslant\cdots\geqslant s_{n-2}\geqslant 0}\left(\sum_{i=0}^{n-2}T_i(\rho)Z^{(i+1)}(s_i)\right)\beta \qquad (3.1)$$

where $Z^{(k)}(t) = \alpha^\vee(\pi(t))$ *if* k *is even and* $Z^{(k)}(t) = \beta^\vee(\pi(t))$ *if* k *is odd. The* $T_k(x)$ *are the Tchebycheff polynomials defined by*

$$T_0(x) = 1, \qquad T_1(x) = 2x, \qquad 2xT_k(x) = T_{k-1}(x) + T_{k+1}(x) \quad \text{for } k \geqslant 1. \tag{3.2}$$

The Tchebycheff polynomials satisfy $T_k(\cos\theta) = \frac{\sin(k+1)\theta}{\sin\theta}$ and, in particular, under the assumptions on ρ and n, $T_k(\rho) \geqslant 0$ for all $k \leqslant n-1$. An important property of the Pitman transforms is the following corollary (see [3]).

Theorem 3.6 *(Generalized braid relations for the Pitman transforms). Let* $\alpha, \beta \in V$ *and* $\alpha^\vee, \beta^\vee \in V^\vee$ *be such that* $\alpha^\vee(\alpha) = \beta^\vee(\beta) = 2$, *and* $\alpha^\vee(\beta) < 0$, $\beta^\vee(\alpha) < 0$ *and* $\alpha^\vee(\beta)\beta^\vee(\alpha) = 4\cos^2\frac{\pi}{n}$, *where* $n \geqslant 2$ *is some integer. Then*

$$\mathcal{P}_\alpha\mathcal{P}_\beta\mathcal{P}_\alpha \ldots = \mathcal{P}_\beta\mathcal{P}_\alpha\mathcal{P}_\beta \ldots$$

where there are n factors in each product.

3.4. *Pitman transforms for Coxeter groups*

Let (W, S) be a Coxeter system, with a realization in the space V. For a simple reflection s, denote by $\mathcal{P}_{\alpha_s}$ or $\mathcal{P}_s$ the Pitman transform associated with the pair $(\alpha_s, \alpha_s^\vee)$. From Theorem 3.6 and Matsumoto's lemma [4, Ch. IV, No. 1.5, Prop. 5], we deduce [3]:

Theorem 3.7. *Let* $w = s_1 \ldots s_r$ *be a reduced decomposition of* $w \in W$, *with* $s_1, \ldots, s_r \in S$. *Then*

$$\mathcal{P}_w := \mathcal{P}_{s_1} \ldots \mathcal{P}_{s_r}$$

depends only on w *and not on the chosen decomposition.*

When W is finite, it has a unique longest element, denoted by w_0. The transformation $\mathcal{P}_{w_0}$ plays a fundamental role in the sequel. The following result is proved in [3].

Proposition 3.8. *When* W *is finite, for any path* $\eta \in C_T^0(V)$, *the path* $\mathcal{P}_{w_0}\eta$ *takes values in the closed Weyl chamber* $\overline{C}$. *Furthermore* $\mathcal{P}_{w_0}$ *is an idempotent and* $\mathcal{P}_w\mathcal{P}_{w_0} = \mathcal{P}_{w_0}\mathcal{P}_w = \mathcal{P}_{w_0}$ *for all* $w \in W$.

3.5. *The continuous crystal* $C_T^0(V)$

For any path η in $C_T^0(V)$, let $wt(\eta) = \eta(T)$. Let e_α^r be the generalized Littelmann operator $\mathcal{E}_\alpha^r$ defined in Definition 3.3, and

$$\varepsilon_\alpha(\eta) = \max\{r \geqslant 0;\ \mathcal{E}_\alpha^r(\eta) \neq 0\} = -\inf_{0\leqslant t\leqslant T} \alpha^\vee\big(\eta(t)\big),$$

$$\varphi_\alpha(\eta) = \max\{r \geqslant 0;\ \mathcal{E}_\alpha^{-r}(\eta) \neq 0\} = \alpha^\vee\big(\eta(T)\big) - \inf_{0\leqslant t\leqslant T} \alpha^\vee\big(\eta(t)\big).$$

It is clear that

Proposition 3.9. *With the above definitions, $C_T^0(V)$ is a normal continuous crystal.*

We say that a path is dominant if it takes its values in the closed Weyl chamber $\overline{C}$.

Definition 3.10. Let $\pi \in C_T^0(V)$ be a dominant path, and $w \in W$. We define

$$L_\pi^w = \{\eta \in C_T^0(V);\ \mathcal{P}_w \eta = \pi\}.$$

These sets are defined for arbitrary Coxeter groups. We shall establish their main properties in the case of finite Coxeter groups, where they are analogues of Demazure–Littelmann modules. It remains an interesting problem to establish similar properties in the general case.

From now on we assume that W is finite, with longest element w_0, and we denote $L_\pi = L_\pi^{w_0}$, which we call the Littelmann module associated with π. The set $L_\pi \cup \{\mathbf{0}\}$ is a subset of $C_T^0(V) \cup \{\mathbf{0}\}$ invariant under the Littelmann operators, thus:

Theorem 3.11. *For any dominant path π, L_π is a normal continuous crystal with highest weight $\pi(T)$.*

Proof. This follows from the result of Section 3.4, except the highest weight property, which follows from the fact that, see (3.5), any $\eta \in L_\pi$ can be written as

$$\eta = \mathcal{H}_{s_q}^{x_q} \mathcal{H}_{s_{q-1}}^{x_{q-1}} \cdots \mathcal{H}_{s_1}^{x_1} \pi. \qquad \square$$

Two paths η_1 and η_2 are said to be connected if there exists simple roots $\alpha_1, \ldots, \alpha_k$ and real numbers $r_1, \ldots, r_k$ such that

$$\eta_1 = \mathcal{E}_{\alpha_1}^{r_1} \cdots \mathcal{E}_{\alpha_k}^{r_k} \eta_2.$$

This is equivalent with the relation $\mathcal{P}_{w_0}\eta_1 = \mathcal{P}_{w_0}\eta_2$. A connected set in $C_T^0(V)$ is a subset in which each two elements are connected. We see that the sets $\{L_\pi, \pi \text{ dominant}\}$ are the connected components in $C_T^0(V)$. Moreover we will show in Theorem 4.9 that the continuous crystals L_{π_1} and L_{π_2} are isomorphic if and only if $\pi_1(T) = \pi_2(T)$.

3.6. Braid relations for the $\mathcal{H}$ operators

Let $w \in W$ and fix a reduced decomposition $w = s_1 \ldots s_p$. For any path η in $C_T^0(V)$, denote $\eta_p = \eta$ and for $k = 1, \ldots, p$,

$$\eta_{k-1} = \mathcal{P}_{s_k} \ldots \mathcal{P}_{s_p} \eta.$$

Then $\eta_{k-1} = \mathcal{P}_{s_k}\eta_k$ is α_{s_k}-dominant, by Proposition 3.2(ii) and

$$\eta_k = \mathcal{F}_{s_k}^{x_k} \eta_{k-1} = \mathcal{H}_{s_k}^{x_k} \eta_{k-1}$$

where

$$x_k = -\inf_{0 \leqslant t \leqslant T} \alpha_{s_k}^\vee\big(\eta_k(t)\big). \tag{3.3}$$

Observe that

$$x_k \in \big[0, \alpha_{s_k}^\vee\big(\eta_{k-1}(T)\big)\big] \tag{3.4}$$

and

$$\eta_k(T) = \eta_{k-1}(T) - x_k \alpha_{s_k};$$

thus,

$$\eta_k(T) = \eta_0(T) - \sum_{i=1}^{k} x_i \alpha_{s_i}.$$

Furthermore,

$$\eta_k = \mathcal{H}_{s_k}^{x_k} \mathcal{H}_{s_{k-1}}^{x_{k-1}} \cdots \mathcal{H}_{s_1}^{x_1} \mathcal{P}_w \eta, \tag{3.5}$$

and the numbers $(x_1, \ldots, x_k)$ are uniquely determined by this equation.

We consider two reduced decompositions

$$w = s_1 \ldots s_p, \qquad w = s_1' \ldots s_p'$$

of w. Let $\mathbf{i} = (s_1, \ldots, s_p)$ and $\mathbf{j} = (s_1', \ldots, s_p')$. Let $\eta : [0, T] \to V$ be a continuous path such that $\eta(0) = 0$, and let $(x_1, \ldots, x_p)$, respectively $(y_1, \ldots, y_p)$, be the numbers determined by Eq. (3.5) for the two decompositions $\mathbf{i}$ and $\mathbf{j}$. The following theorem states that the correspondence between the x_n's and the y_n's actually does not depend on the path η. In other words, we have the following braid relation for the operators $\mathcal{H}$:

$$\mathcal{H}_{s_p}^{x_p} \cdots \mathcal{H}_{s_2}^{x_2} \mathcal{H}_{s_1}^{x_1} = \mathcal{H}_{s_p'}^{y_p} \cdots \mathcal{H}_{s_2'}^{y_2} \mathcal{H}_{s_1'}^{y_1}. \tag{3.6}$$

Theorem 3.12. *There exists a piecewise linear continuous map $\phi_{\mathbf{i}}^{\mathbf{j}} : \mathbb{R}^p \to \mathbb{R}^p$ such that for all paths $\eta \in C_T^0(V)$,*

$$(y_1, \ldots, y_p) = \phi_{\mathbf{i}}^{\mathbf{j}}(x_1, \ldots, x_p).$$

Proof.

First step. If the roots α, β generate a system of type $A_1 \times A_1$ and $w = s_\alpha s_\beta = s_\beta s_\alpha$, then $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$ commute, and it is immediate that $x_1 = y_2$, $x_2 = y_1$. Let $\alpha, \alpha^\vee$ and $\beta, \beta^\vee$ be such that

$$\alpha^\vee(\alpha) = \beta^\vee(\beta) = 2, \qquad \alpha^\vee(\beta) = \beta^\vee(\alpha) = -1,$$

then α and β generate a root system of type A_2 and the braid relation is

$$w_0 = s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta.$$

Define

$$a \wedge b = \min(a, b), \qquad a \vee b = \max(a, b).$$

We prove that the following map

$$\begin{aligned} x_1 &= (y_2 - y_1) \wedge y_3, & y_1 &= (x_2 - x_1) \wedge x_3, \\ x_2 &= y_1 + y_3, & y_2 &= x_1 + x_3, \\ x_3 &= y_1 \vee (y_2 - y_3), & y_3 &= x_1 \vee (x_2 - x_3) \end{aligned} \tag{3.7}$$

satisfies the required properties. Assume that, for $\pi = \mathcal{P}_{w_0}\eta$,

$$\eta = \mathcal{H}_\alpha^{x_3}\mathcal{H}_\beta^{x_2}\mathcal{H}_\alpha^{x_1}\pi.$$

Then define $\eta_2 = \mathcal{P}_\alpha\eta$, $\eta_1 = \mathcal{P}_\beta\mathcal{P}_\alpha\eta$, $\eta_0 = \pi = \mathcal{P}_\alpha\mathcal{P}_\beta\mathcal{P}_\alpha\eta$. Using Theorem 3.5 for computing the paths η_i one gets the explicit formulas:

$$\begin{aligned} x_3 &= -\inf_{0\leqslant s\leqslant T} \alpha^\vee\big(\eta(s)\big), \\ x_2 &= -\inf_{0\leqslant s_2\leqslant s_1\leqslant T} \big(\beta^\vee\big(\eta(s_1)\big) + \alpha^\vee\big(\eta(s_2)\big)\big), \\ x_1 &= -\inf_{0\leqslant s_2\leqslant s_1\leqslant T} \big(\alpha^\vee\big(\eta(s_1)\big) + \beta^\vee\big(\eta(s_2)\big)\big) - x_3. \end{aligned}$$

Similar formulas are obtained for the y_i coming from the other reduced decomposition, by exchanging the roles of α and β. The formula (3.7) follows by inspection.

In the context of crystals, this result is well known and first appeared in Lusztig [25] and Kashiwara [20]. We observe that it can also be obtained from the considerations of Section 6, see, e.g. Section 6.7.

Second step. When the roots generate a root system of type A_n, using Matsumoto's lemma, one can pass from one reduced decomposition to another by a sequence of braid relations corresponding to the two cases of the first step.

Third step. We consider now the case where the roots generate the dihedral group $I(m)$, and $w = s_\alpha s_\beta \ldots = s_\beta s_\alpha \ldots$ is the longest element in W. We will use an embedding of the dihedral group $I(m)$ in the Weyl group of the system A_{m-1}, see e.g. Bourbaki [4, Ch. V, 6, Lemme 2]. Recall the Tchebycheff polynomials T_k defined in (3.2). Let $\lambda = \cos(2\pi/m)$, $a_1 = a_2 = 1$ and, for $k \geqslant 1$,

$$a_{2k} = T_{k-1}(\lambda), \qquad a_{2k+1} = T_k(\lambda) + T_{k-1}(\lambda)$$

then,

$$a_{2k} + a_{2k+2} = a_{2k+1}, \qquad a_{2k+1}a_{2k-1} + a_{2k+1} = (1 + a_3)a_{2k}. \tag{3.8}$$

Moreover $a_k > 0$ when $k < m$ and $a_m = 0$.

In the Euclidean space $V = \mathbb{R}^{m-1}$ we choose simple roots $\alpha_1, \dots, \alpha_{m-1}$ which satisfy $\langle \alpha_i, \alpha_j \rangle = a_{ij}$ where $a_{ij} = 2$ if $i = j$, $a_{ij} = -1$ if $|i - j| = 1$, $a_{ij} = 0$ otherwise. Let $\alpha_i^\vee = \alpha_i$ and $s_i = s_{\alpha_i}$. These generate a root system of type A_{m-1}.

Let Π be the two-dimensional plane defined as the set of $x \in V$ such that for all $n < m$,

$$\langle \alpha_n, x \rangle = a_n \langle \alpha_1, x \rangle$$

if n is odd, and

$$\langle \alpha_n, x \rangle = a_n \langle \alpha_2, x \rangle$$

if n is even. It follows from the relation (3.8) that the vectors

$$\alpha = \sum_{n \text{ odd},\ n<m} a_n \alpha_n, \qquad \beta = \sum_{n \text{ even},\ n<m} a_n \alpha_n$$

are in Π. Let $\alpha^\vee = 2\alpha/\|\alpha\|^2$, $\beta^\vee = 2\beta/\|\beta\|^2$ and

$$\tau_1 = s_1 s_3 s_5 \dots s_{2p-1},$$
$$\tau_2 = s_2 s_4 s_6 \dots s_{2r},$$

where $2p = m - 1, r = p$ when m is odd and $2p = m, r = p - 1$ when m is even. Let w_0 be the longest element in the Weyl group of A_{m-1}. Its length is $q = (m - 1)m/2$. We first consider the case where m is odd, $m = 2p + 1, q = pm$. Then

$$w_0 = (\tau_1 \tau_2)^p \tau_1, \quad \text{and} \quad w_0 = \tau_2 (\tau_1 \tau_2)^p$$

are two reduced decompositions of w_0. Since $(\tau_1 \tau_2)^m = Id$ the angle between α and $-\beta$ is π/m and these vectors are the simple roots of the dihedral system $I(m)$.

Let γ be a continuous path in Π, let $\gamma_p = \gamma$ and for $1 < k \leqslant p$, $\gamma_{k-1} = \mathcal{P}_{\alpha_{2k-1}} \gamma_k$ and

$$z_k(t) = - \inf_{0 \leqslant s \leqslant t} \alpha_{2k-1}^\vee \big(\gamma_k(s)\big).$$

Lemma 3.13. *Let γ be a continuous path with values in Π and let*

$$x(t) = - \inf_{0 \leqslant s \leqslant t} \alpha^\vee \big(\gamma(s)\big).$$

Then, for all k, $z_k(t) = a_{2k-1} x(t)$ and

$$\mathcal{P}_{\tau_1} \gamma(t) = \mathcal{P}_{\alpha_1} \mathcal{P}_{\alpha_3} \mathcal{P}_{\alpha_5} \dots \mathcal{P}_{\alpha_{2p-1}} \gamma(t) = \gamma(t) - \inf_{s \leqslant t} \alpha^\vee \big(\gamma(s)\big) \alpha = \mathcal{P}_\alpha \gamma(t).$$

Proof. First, notice that $\alpha^\vee(\gamma(t)) = \alpha_1^\vee(\gamma(t))$. Since γ is in Π, one has

$$z_p(t) = - \inf_{0 \leqslant s \leqslant t} \alpha_{2p-1}^\vee \big(\gamma(s)\big) = - \inf_{0 \leqslant s \leqslant t} a_{2p-1} \alpha_1^\vee \big(\gamma(s)\big) = a_{2p-1} x(t)$$

where we use the positivity of a_{2p-1}. Therefore

$$\gamma_{p-1}(t) = \mathcal{P}_{\alpha_{2p-1}}\gamma(t) = \gamma(t) + z_p(t)\alpha_{2p-1} = \gamma(t) + a_{2p-1}x(t)\alpha_{2p-1}.$$

Now, since the α_{2i+1} are orthogonal,

$$z_{p-1}(t) = -\inf_{0\leqslant s\leqslant t}\alpha^\vee_{2p-3}\big(\gamma_{p-1}(s)\big) = -\inf_{0\leqslant s\leqslant t}\alpha^\vee_{2p-3}\big(\gamma(s)\big) = a_{2p-3}x(t),$$

and

$$\begin{aligned}\gamma_{p-2}(t) &= \mathcal{P}_{\alpha_{2p-3}}\gamma_{p-1}(t) = \gamma_{p-1}(t) + z_{p-1}(t)\alpha_{2p-3}\\ &= \gamma(t) + x(t)(a_{2p-3}\alpha_{2p-3} + a_{2p-1}\alpha_{2p-1}).\end{aligned}$$

Continuing, we obtain that

$$z_k(t) = a_{2k-1}x(t),$$
$$\gamma_k(t) = \gamma(t) + x(t)(a_{2k-1}\alpha_{2k-1} + \cdots + a_{2p-1}\alpha_{2p-1}).$$

Since $\alpha = \alpha_1 + a_3\alpha_3 + a_5\alpha_5 + \cdots + a_{2p-1}\alpha_{2p-1}$ we obtain the lemma. □

We have similarly, if γ is a path in Π,

$$\mathcal{P}_{\tau_2}\gamma(t) = \mathcal{P}_{\alpha_2}\mathcal{P}_{\alpha_4}\mathcal{P}_{\alpha_6}\dots\mathcal{P}_{\alpha_{2r}}\gamma(t) = \gamma(t) - \inf_{s\leqslant t}\beta^\vee\big(\gamma(s)\big)\beta = \mathcal{P}_\beta\gamma(t).$$

Let $\mathbf{i} = (s_{i_1},\dots,s_{i_q}) = (\mathbf{i}_1,\mathbf{i}_2,\dots,\mathbf{i}_m)$ and $\mathbf{j} = (s_{j_1},\dots,s_{j_q}) = (\mathbf{j}_1,\mathbf{j}_2,\dots,\mathbf{j}_m)$ where $\mathbf{i}_k = \mathbf{j}_{k+1} = (s_1,s_3,\dots,s_{2p-1})$ when k is odd and $\mathbf{i}_k = \mathbf{j}_{k+1} = (s_2,s_4,\dots,s_{2p})$ when k is even. We write explicitly

$$w_0 = (\tau_1\tau_2)^p\tau_1 = s_{i_1}\dots s_{i_q}, \qquad w_0 = \tau_2(\tau_1\tau_2)^p = s_{j_1}\dots s_{j_q}.$$

Let us denote by $\phi_{\mathbf{i}}^{\mathbf{j}} : \mathbb{R}^q \to \mathbb{R}^q$ the mapping given by the second step corresponding to these two reduced decompositions of w_0 in the Weyl group of A_{m-1}.

Let γ be a path with values in Π. If we consider it as a path in V we can set $\eta_q = \tilde\eta_q = \gamma$ and, for $n = 1,2,\dots,q$,

$$\eta_{n-1} = \mathcal{P}_{\alpha_{i_n}}\eta_n, \qquad z_n = -\inf_{0\leqslant t\leqslant T}\alpha^\vee_{i_n}\big(\eta_n(t)\big),$$
$$\tilde\eta_{n-1} = \mathcal{P}_{\alpha_{j_n}}\tilde\eta_n, \qquad \tilde z_n = -\inf_{0\leqslant t\leqslant T}\alpha^\vee_{j_n}\big(\tilde\eta_n(t)\big).$$

Then, by definition,

$$(\tilde z_1,\dots,\tilde z_q) = \phi_{\mathbf{i}}^{\mathbf{j}}(z_1,\dots,z_q).$$

We now consider γ as a path in Π. We let

$$(u_1, u_2, \dots, u_m) = (\alpha, \beta, \alpha, \beta, \dots, \alpha)$$

and

$$(v_1, v_2, \dots, v_m) = (\beta, \alpha, \beta, \alpha, \dots, \beta).$$

In $I(m)$ the two reduced decompositions of the longest element are

$$s_{u_1} \dots s_{u_m} = s_{v_1} \dots s_{v_m}.$$

We introduce $\gamma_m = \tilde{\gamma}_m = \gamma$, and, for $n = 1, 2, \dots, m$,

$$\gamma_{n-1} = \mathcal{P}_{u_n} \dots \mathcal{P}_{u_m} \gamma_m, \qquad \tilde{\gamma}_{n-1} = \mathcal{P}_{v_n} \dots \mathcal{P}_{v_m} \tilde{\gamma}_m,$$
$$x_n = -\inf_{0 \leqslant t \leqslant T} u_n^\vee\big(\gamma_n(t)\big), \qquad \tilde{x}_n = -\inf_{0 \leqslant t \leqslant T} v_n^\vee\big(\tilde{\gamma}_n(t)\big).$$

It follows from Lemma 3.13 and from its analogue with α replaced by β that

$$z_1 = a_1 x_1, \quad z_2 = a_3 x_1, \quad \dots, \quad z_p = a_{2p-1} x_1,$$
$$z_{p+1} = a_2 x_2, \quad z_{p+2} = a_4 x_2, \quad \dots, \quad z_{2p} = a_{2p} x_2,$$

and more generally, for $k = 0, \dots$

$$a_1^{-1} z_{2kp+1} = a_3^{-1} z_{2kp+2} = \dots = a_{2p-1}^{-1} z_{2kp+p} = x_{k+1},$$
$$a_2^{-1} z_{(2k+1)p+1} = a_4^{-1} z_{(2k+1)p+2} = \dots = a_{2p}^{-1} z_{(2k+2)p} = x_{k+2}.$$

This defines a linear map

$$(x_1, \dots, x_m) = g(z_1, z_2, \dots, z_q).$$

Analogously exchanging the role of α and β we define a similar map

$$(\tilde{x}_1, \dots, \tilde{x}_m) = \tilde{g}(\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_q)$$

(for instance $\tilde{z}_1 = a_2 \tilde{x}_1, \tilde{z}_2 = a_3 \tilde{x}_1, \dots$). Then we see that

$$(x_1, \dots, x_m) = \phi(\tilde{x}_1, \dots, \tilde{x}_m)$$

where $\phi = \tilde{g} \circ \phi_{\mathbf{i}}^{\mathbf{j}} \circ g^{-1}$. The proof when m is even is similar (when $m = 2p$, $w_0 = (\tau_1\tau_2)^p$ and $w_0 = (\tau_2\tau_1)^p$ are two reduced decompositions of w_0). This proves the theorem in the dihedral case.

Fourth step. We use Matsumoto's lemma to reduce the general case to the dihedral case.

This ends the proof of Theorem 3.12. □

Remark 3.14. Although the given proof is constructive, it gives a complicated expression for $\phi_{\mathbf{i}}^{\mathbf{j}}$ which can sometimes be simplified. In the dihedral case $I(m)$, for the Weyl group case, i.e. $m = 3, 4, 6$, these expressions are given in Littelmann [24]. For $m = 5$ it can be shown by a tedious verification that it is given when α, β have the same length, by a similar formula. Thus for $m = 2, 3, 4, 5, 6$ let $c_0 = 1$, $c_1 = 2\cos(\pi/m)$, $c_{n+1} + c_{n-1} = c_1 c_n$ for $n \geqslant 0$, and

$$u = \max(c_k x_{k+1} - c_{k-1} x_{k+2},\ 0 \leqslant k \leqslant m-3),$$
$$v = \min(c_k x_{k+2} - c_{k+1} x_{k+1},\ 1 \leqslant k \leqslant m-2).$$

Then the expressions are given by

$$y_m = \max(x_{m-1} - c_1 x_m, u),$$
$$y_{m-1} = x_m + \max(x_{m-2} - c_2 x_m, c_1 u),$$
$$y_2 = x_1 + \min(x_3 - c_2 x_1, c_1 v),$$
$$y_1 = \min(x_2 - c_1 x_1, v)$$

and

$$y_1 + y_3 + \cdots = x_2 + x_4 + \cdots,$$
$$y_2 + y_4 + \cdots = x_1 + x_3 + \cdots.$$

This determines completely $(y_1, \ldots, y_m)$ as a function of $(x_1, \ldots, x_m)$ when $m \leqslant 6$. For $m = 7$ we think (and made a computer check) that we have to add that

$$y_7 + y_5 = x_6 + \max(c_2 x_1, x_4 - c_3 x_7, w),$$
$$w = \min\bigl(c_2 u, x_4 - c_2 v, \max(x_6 - c_1 x_5 + x_4 + c_2 u, c_1 x_3 - x_2 - c_2 v)\bigr).$$

We do not know of similar formulas for $m \geqslant 8$.

Remark 3.15. The map given by Theorem 3.12 is unique on the set of all possible coordinates of paths. We will see in the next section that this set is a convex cone. Since the value of the map $\phi_{\mathbf{i}}^{\mathbf{j}}$ is irrelevant outside this cone, we may say that there exists a unique such map for each pair of reduced decompositions $\mathbf{i}$, $\mathbf{j}$.

4. Parametrization of the continuous Littelmann module

In this section we make a more in-depth study of the parametrization of the Littelmann modules, and we prove the analogue of the independence theorem of Littelmann (the crystal structure depends only on the endpoint of the dominant path), then we study the concatenation of paths, using it to prove existence and uniqueness of families of crystals. Finally we define the action of the Coxeter group on the crystal, and the Schützenberger involution.

4.1. String parametrization of $C_T^0(V)$

Let $(W, S, V, V^\vee)$ be a realization of the Coxeter system (W, S). From now on we assume that W is finite, with longest element w_0. For notational convenience, we sometimes write $\alpha^\vee\eta$ instead of $\alpha^\vee(\eta)$.

Let $\eta \in L_\pi$, where π is dominant and $w_0 = s_1 \dots s_q$ be a reduced decomposition, then we have seen that

$$\eta = \mathcal{H}_{s_q}^{x_q} \mathcal{H}_{s_{q-1}}^{x_{q-1}} \cdots \mathcal{H}_{s_1}^{x_1} \pi$$

for a unique sequence

$$\varrho_{\mathbf{i}}(\eta) = (x_1, \dots, x_q).$$

Following Berenstein and Zelevinsky [2], we call $\varrho_{\mathbf{i}}(\eta)$ the **i**-string parametrization of η, or the string parametrization if no confusion is possible.

We let

$$C_{\mathbf{i}}^\pi = \varrho_{\mathbf{i}}(L_\pi),$$

this is the set of all the $(x_1, \dots, x_q) \in \mathbb{R}^q$ which occur in the string parametrizations of the elements of L_π.

Proposition 4.1. *The set L_π is compact and the map $\varrho_{\mathbf{i}}$ is a bicontinuous bijection from L_π onto its image $C_{\mathbf{i}}^\pi$.*

Proof. The map $\varrho_{\mathbf{i}}$ has an inverse

$$\varrho_{\mathbf{i}}^{-1}(x_1, \dots, x_q) = \mathcal{H}_{s_q}^{x_q} \mathcal{H}_{s_{q-1}}^{x_{q-1}} \cdots \mathcal{H}_{s_1}^{x_1} \pi,$$

hence it is bijective. It is clear that $\varrho_{\mathbf{i}}$ and $\varrho_{\mathbf{i}}^{-1}$ are continuous. Since $\mathcal{P}_{w_0}$ is continuous, $L_\pi = \{\eta;\ \mathcal{P}_{w_0}(\eta) = \pi\}$ is closed. Using $\varrho_{\mathbf{i}}^{-1}$ we easily see that L_π is equicontinuous, it is thus compact by Ascoli's theorem. □

We will study $C_{\mathbf{i}}^\pi$ in detail in the following sections.

4.2. The crystallographic case

In this subsection we consider the case of a Weyl group W with a crystallographic root system. When α is a root and $\alpha^\vee$ its coroot, then $\mathcal{E}_\alpha^1$ and $\mathcal{E}_\alpha^{-1}$ from Definition 3.3 coincide with the Littelmann operators e_α and f_α, defined in [23]. Recall that a path η is called integral in [23] if its endpoint $\eta(T)$ is in the weight lattice and if, for each simple root α, the minimum of the function $\alpha^\vee(\eta(t))$ over $[0, T]$ is an integer. The class of integral paths is invariant under the Littelmann operators.

Let π be a dominant integral path. The discrete Littelmann module D_π is defined as the orbit of π under the semigroup generated by all the transformations e_α, f_α, for all simple roots α, so it is the set of integral paths in L_π.

Let $\mathbf{i} = (s_1, \dots, s_q)$ where $w_0 = s_1 \dots s_q$ is a reduced decomposition, then it follows from Littelmann's theory that

$$D_\pi = \{\eta \in L_\pi;\ x_1, \dots, x_q \in \mathbb{N}\} = \varrho_{\mathbf{i}}^{-1}\big(\big\{(x_1, \dots, x_q) \in C_{\mathbf{i}}^\pi;\ x_1 \in \mathbb{N}, \dots, x_q \in \mathbb{N}\big\}\big).$$

Furthermore, the set D_π has a crystal structure isomorphic to the Kashiwara crystal associated with the highest weight $\pi(T)$. On D_π the coordinates $(x_1, \dots, x_q)$ are called the string or the Kashiwara parametrization of the dual canonical basis. They are described in Littelmann [24] and Berenstein and Zelevinsky [2].

When restricted to D_π, the Pitman operator $\mathcal{P}_\alpha$ coincides with $e_\alpha^{\max}$, i.e. the operator sending η to $e_\alpha^n \eta$, where $n = \max(k, e_\alpha^k \eta \neq \mathbf{0})$.

For any path $\eta : [0, T] \to V$ and $\lambda > 0$ let $\lambda\eta$ be the path defined by $(\lambda\eta)(t) = \lambda\eta(t)$ for $0 \leqslant t \leqslant T$. The following results are immediate.

Proposition 4.2 *(Scaling property).*

(i) *For any $\lambda > 0$, $\lambda L_\pi = L_{\lambda\pi}$.*
(ii) *Let $\eta \in C_T^0(V)$, $r \in \mathbb{R}$, $u > 0$, then $\mathcal{E}_\alpha^{ru}(u\eta) = u\mathcal{E}_\alpha^r(\eta)$.*
(iii) *Let π be a dominant path and $a > 0$ then $C_{\mathbf{i}}^{a\pi} = aC_{\mathbf{i}}^\pi$.*

Proposition 4.3. *If π is a dominant integral path, then the set*

$$D_\pi(\mathbb{Q}) = \bigcup_{n \in \mathbb{N}} \frac{1}{n} D_{n\pi}$$

is dense in L_π.

Actually a good interpretation of L_π in the Weyl group case is as the "limit" of $\frac{1}{n} B_{n\pi}$ when $n \to \infty$. In the general Coxeter case only the limiting object is defined.

4.3. Polyhedral nature of the continuous crystal for a Weyl group

Let W be a finite Weyl group, associated to a crystallographic root system. Let D_π be the discrete Littelmann module associated with an integral dominant path π. We fix a reduced decomposition $w_0 = s_1 \dots s_q$ of the longest element and let $\mathbf{i} = (s_1, \dots, s_q)$. We have seen that if $\rho_{\mathbf{i}} : L_\pi \to C_{\mathbf{i}}^\pi$ is the string parametrization of the continuous module L_π, then

$$D_\pi = \{\eta \in L_\pi;\ x_1, \dots, x_q \in \mathbb{N}\} = \varrho_{\mathbf{i}}^{-1}\big(\big\{(x_1, \dots, x_q) \in C_{\mathbf{i}}^\pi;\ x_1 \in \mathbb{N}, \dots, x_q \in \mathbb{N}\big\}\big).$$

Therefore the set

$$\tilde{C}_{\mathbf{i}}^\pi = C_{\mathbf{i}}^\pi \cap \mathbb{N}^q$$

is the image of the discrete Littelmann module D_π, or equivalently, the image of the Kashiwara crystal with highest weight $\pi(T)$, under the string parametrization of Littelmann [24] and Berenstein and Zelevinsky [2]. Let

$$K_\pi = \left\{(x_1, \ldots, x_q) \in \mathbb{R}^q;\ 0 \leqslant x_r \leqslant \alpha_{i_r}^\vee\left(\pi(T) - \sum_{n=1}^{r-1} x_n \alpha_{i_n}\right),\ r = 1, \ldots, q\right\}.$$

It is shown in Littelmann [24] that there exists a convex rational polyhedral cone $C_{\mathbf{i}}$ in $\mathbb{R}^q$, depending only on $\mathbf{i}$ such that, for all dominant integral paths π,

$$\tilde{C}_{\mathbf{i}}^\pi = C_{\mathbf{i}} \cap \mathbb{N}^q \cap K_\pi.$$

This cone is described explicitly in Berenstein and Zelevinsky [2]. Recall that $C_{\mathbf{i}}^\pi = \varrho_{\mathbf{i}}(L_\pi)$. Using Propositions 4.2, 4.3 it is easy to see that the following holds.

Proposition 4.4. *For all dominant paths π, $C_{\mathbf{i}}^\pi = C_{\mathbf{i}} \cap K_\pi$.*

4.4. The cone in the general case

We now consider a general Coxeter system (W, S), with W finite, realized in V.

Theorem 4.5. *Let $\mathbf{i}$ be a reduced decomposition of w_0, then there exists a unique polyhedral cone $C_{\mathbf{i}}$ in $\mathbb{R}^q$ such that for any dominant path π*

$$C_{\mathbf{i}}^\pi = C_{\mathbf{i}} \cap K_\pi.$$

In particular $C_{\mathbf{i}}^\pi$ depends only on $\lambda = \pi(T)$.

Proof. It remains to consider the non-crystallographic Coxeter systems. It is clearly enough to consider reduced systems. We use their classification: W is either a dihedral group $I(m)$ or H_3 or H_4 (see Humphreys [16]), and the same trick as the one used in the proof of Theorem 3.12.

We first consider the case $I(m)$ where $m = 2p + 1$ and we use the notation of the proof of Theorem 3.12. Let $\mathbf{i} = (i_1, \ldots, i_q)$ be as in that proof, and write

$$w_0 = (\tau_1\tau_2)^p \tau_1 = s_{i_1} \ldots s_{i_q}$$

for the longest word in A_{m-1}. Let γ be a path with values in the plane Π. If we consider γ as a path in $V = \mathbb{R}^{m-1}$ we can set, for $q = (m-1)m/2$, $\eta_q = \gamma$ and, for $n = 1, 2, \ldots, q$,

$$\eta_{n-1} = \mathcal{P}_{\alpha_{i_n}} \eta_n, \qquad z_n = -\inf_{0 \leqslant t \leqslant T} \alpha_{i_n}^\vee\big(\eta_n(t)\big).$$

We can also consider γ as a path in Π, with the realization of $I(m)$. Let

$$\mathbf{u} = (u_1, u_2, \ldots, u_m) = (\alpha, \beta, \alpha, \beta, \ldots, \alpha).$$

Let $\tilde{\eta}_m = \gamma$ and, for $n = 1, 2, \dots, m$,

$$\tilde{\eta}_{n-1} = \mathcal{P}_{u_n} \dots \mathcal{P}_{u_m} \eta_m, \qquad x_n = -\inf_{0 \leqslant t \leqslant T} u_n^\vee\bigl(\eta_n(t)\bigr).$$

We have seen that the map

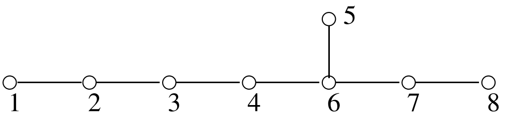
$$(x_1, \dots, x_m) = g(z_1, z_2, \dots, z_q),$$

is linear. Let $C_{\mathbf{i}}$ be the cone associated with $\mathbf{i}$ in A_{m-1}, then $C_{\mathbf{u}} = g(C_{\mathbf{i}})$ is the cone in $\mathbb{R}^m$ associated with the reduced decomposition $\alpha\beta \dots \alpha$ of the longest word in $I(m)$. Furthermore, for any dominant path π in Π, $C_{\mathbf{u}}^\pi = C_{\mathbf{u}} \cap K_\pi$.

The proof when m is even is similar.

In order to deal with the cases H_3 and H_4 it is enough, using an analogous proof to embed these systems in some Weyl groups.

Let us first consider the case of H_4. We use the embedding of H_4 in E_8 (see [26]). Consider the following indexation of the simple roots of the system E_8:



System E_8

In the Euclidean space $V = \mathbb{R}^8$ the roots $\alpha_1, \dots, \alpha_8$, satisfy $\langle \alpha_i, \alpha_j \rangle = -1$ or 0 depending whether they are linked or not. Let $\phi = (1 + \sqrt{5})/2$. We consider the 4-dimensional subspace Π of V defined as the set of $x \in V$ orthogonal to $\alpha_8 - \phi\alpha_1, \alpha_7 - \phi\alpha_2, \alpha_6 - \phi\alpha_3$ and $\phi\alpha_5 - \alpha_4$. Let s_i be the reflection which corresponds to α_i and

$$\tau_1 = s_1 s_8, \qquad \tau_2 = s_2 s_7, \qquad \tau_3 = s_3 s_6, \qquad \tau_4 = s_4 s_5.$$

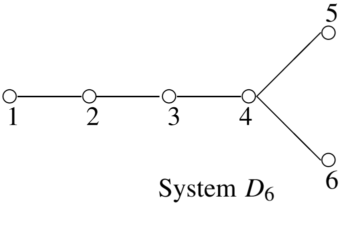
One checks easily that $\tau_1, \tau_2, \tau_3, \tau_4$ generate H_4 and that the vectors

$$\tilde{\alpha}_1 = \alpha_1 + \phi\alpha_8, \qquad \tilde{\alpha}_2 = \alpha_2 + \phi\alpha_7, \qquad \tilde{\alpha}_3 = \alpha_3 + \phi\alpha_6, \qquad \tilde{\alpha}_4 = \alpha_4 + \phi^{-1}\alpha_5$$

are in Π. If π is a continuous path in Π, then, for $i = 1, \dots, 4$, if $\tilde{\alpha}_i^\vee = \tilde{\alpha}_i / (2\|\tilde{\alpha}_i\|^2)$,

$$\mathcal{P}_{\tau_i}\pi(t) = \pi(t) - \inf_{0 \leqslant s \leqslant t} \tilde{\alpha}_i^\vee\bigl(\pi(s)\bigr)\tilde{\alpha}_i.$$

The case of H_3 is similar by using D_6:



System D_6

In $V=\mathbb{R}^6$ we choose the roots $\alpha_1,\dots,\alpha_6$ with $\langle\alpha_i,\alpha_j\rangle=-1$ if they are linked. We define a 3-dimensional subspace Π defined as the set of $x\in V$ orthogonal to $\alpha_5-\phi\alpha_1,\alpha_4-\phi\alpha_2$ and $\phi\alpha_6-\alpha_3$. Then the reflections

$$\tau_1=s_1s_5,\qquad \tau_2=s_2s_4,\qquad \tau_3=s_3s_6, \tag{4.1}$$

generate H_3 and

$$\tilde\alpha_1=\alpha_1+a\alpha_5,\qquad \tilde\alpha_2=\alpha_2+a\alpha_4,\qquad \tilde\alpha_3=\alpha_3+b\alpha_6$$

are in Π. □

We will prove in Corollary 5.3 that the cones $C_{\mathbf{i}}$ have the following description: for any simple root α, let $\mathbf{j}(\alpha)$ be a reduced decomposition of w_0 which begins by s_α. Then

$$C_{\mathbf{i}}=\bigl\{x\in\mathbb{R}^q;\ \phi_{\mathbf{i}}^{\mathbf{j}(\alpha)}(x)_1\geqslant 0,\ \text{for all simple roots }\alpha\bigr\}.$$

4.5. The cone in the dihedral case

In this section we provide explicit equations for the cone, in the dihedral case, following the approach of Littelmann [24] in the Weyl group case.

Lemma 4.6. *Let $\alpha,\beta\in V$, $\alpha^\vee,\beta^\vee\in V^\vee$ and $c=-\beta^\vee(\alpha)$. Consider a continuous path $\eta\in C_T^0(V)$ and $\pi=\mathcal{P}_\alpha\eta$. Let*

$$\begin{aligned}
U&=\min_{T\geqslant t\geqslant 0}\Bigl[a\beta^\vee\bigl(\eta(t)\bigr)+b\min_{t\geqslant s\geqslant 0}\alpha^\vee\bigl(\eta(s)\bigr)\Bigr],\\
V&=\min_{T\geqslant t\geqslant 0}\Bigl[a\min_{t\geqslant s\geqslant 0}\beta^\vee\bigl(\pi(s)\bigr)+(ac-b)\alpha^\vee\bigl(\pi(t)\bigr)\Bigr],\\
W&=a\min_{T\geqslant t\geqslant 0}\beta^\vee\bigl(\pi(t)\bigr)-(ac-b)\min_{T\geqslant t\geqslant 0}\alpha^\vee\bigl(\eta(t)\bigr),
\end{aligned}$$

where a,b are real numbers such that $a\geqslant 0, ac-b\geqslant 0$. Then $U=\min(V,W)$.

Proof. Since $\pi=\mathcal{P}_\alpha\eta$,

$$\beta^\vee\bigl(\eta(t)\bigr)=\beta^\vee\bigl(\pi(t)\bigr)-c\min_{t\geqslant s\geqslant 0}\alpha^\vee\bigl(\eta(s)\bigr),$$

thus

$$\begin{aligned}
U&=\min_{T\geqslant t\geqslant 0}\Bigl[a\beta^\vee\bigl(\pi(t)\bigr)+(b-ac)\min_{t\geqslant s\geqslant 0}\alpha^\vee\bigl(\eta(s)\bigr)\Bigr]\\
&=\min_{T\geqslant t\geqslant 0}\Bigl[\min_{t\geqslant s\geqslant 0}a\beta^\vee\bigl(\pi(s)\bigr)+(b-ac)\min_{t\geqslant s\geqslant 0}\alpha^\vee\bigl(\eta(s)\bigr)\Bigr]
\end{aligned}$$

where we have used the fact that, if $f, g : [0, T] \to \mathbb{R}$ are two continuous functions, and if g is nondecreasing, then

$$\min_{T \geqslant t \geqslant 0} \big[f(t) + g(t)\big] = \min_{T \geqslant t \geqslant 0} \Big[\min_{t \geqslant s \geqslant 0} f(s) + g(t)\Big].$$

Since $\alpha^\vee(\pi(t)) \geqslant -\min_{t \geqslant s \geqslant 0} \alpha^\vee(\eta(s))$,

$$\min_{t \geqslant s \geqslant 0} a\beta^\vee\big(\pi(s)\big) + (ac-b)\alpha^\vee\big(\pi(t)\big) \geqslant \min_{t \geqslant s \geqslant 0} a\beta^\vee\big(\pi(s)\big) - (ac-b) \min_{t \geqslant s \geqslant 0} \alpha^\vee\big(\eta(s)\big).$$

Let t_0 be the largest $t \leqslant T$ where the minimum of the right-hand side is achieved. Suppose that $t_0 < T$. If $\alpha^\vee(\pi(t_0)) > -\min_{t_0 \geqslant s \geqslant 0} \alpha^\vee(\eta(s))$ then $\min_{t \geqslant s \geqslant 0} \alpha^\vee(\eta(s))$ is locally constant on the right of t_0. Since $\min_{t \geqslant s \geqslant 0} a\beta^\vee(\pi(s))$ is nonincreasing, it follows that t_0 is not maximal. Therefore, when $t_0 < T$,

$$\alpha^\vee\big(\pi(t_0)\big) = -\min_{t_0 \geqslant s \geqslant 0} \alpha^\vee\big(\eta(s)\big)$$

and

$$U = \min_{T \geqslant t \geqslant 0} \Big[\min_{t \geqslant s \geqslant 0} a\beta^\vee\big(\pi(s)\big) - (ac-b) \inf_{t \geqslant s \geqslant 0} \alpha^\vee\big(\eta(s)\big)\Big] = V \leqslant W.$$

When $t_0 = T$, then $U = W \leqslant V$. Thus $U = \min(V, W)$. □

We consider a realization of the dihedral system $I(m)$ with two simple roots α, β and $c := -\alpha^\vee(\beta) = -\beta^\vee(\alpha) = 2\cos\frac{\pi}{m}$. Let

$$a_n = \frac{\sin(n\pi/m)}{\sin(\pi/m)}.$$

Then $a_0 = 0$, $a_1 = 1$, and $a_{n+1} + a_{n-1} = ca_n$, $a_n > 0$ if $1 \leqslant n \leqslant m-1$ and $a_m = 0$. Let $w_0 = s_1 \ldots s_m$ be a reduced decomposition of the longest element $w_0 \in W$, $\mathbf{i} = (s_1, \ldots, s_m)$ and $\alpha_1, \ldots, \alpha_m$ be the simple roots associated with $s_1, \ldots, s_m$. This sequence is either $(\alpha, \beta, \alpha, \ldots)$ or $(\beta, \alpha, \beta, \ldots)$. Clearly the two roots play a symmetric role, and the cones associated with these two decompositions are the same. We define α_0 as the simple root not equal to α_1. As before, when $\eta \in C_T^0(V)$, we define $\eta_m = \eta$ and for $k = 0, \ldots, m-1$, $\eta_k = \mathcal{P}_{s_{k+1}} \ldots \mathcal{P}_{s_m}\eta$, and

$$x_k = -\min_{0 \leqslant t \leqslant T} \alpha_k^\vee\big(\eta_k(t)\big) \quad \text{for } k = 1, \ldots, m.$$

Proposition 4.7. *The cone for the dihedral system $I(m)$ is given by*

$$C_{\mathbf{i}} = \left\{(x_1, \ldots, x_m) \in \mathbb{R}_+^m;\ \frac{x_{m-1}}{a_{m-1}} \geqslant \frac{x_{m-2}}{a_{m-2}} \geqslant \cdots \geqslant \frac{x_1}{a_1}\right\}.$$

Proof. For any p, k such that $0 \leqslant p \leqslant m$, $0 \leqslant k \leqslant p$, let

$$V_k = \min_{T\geqslant t\geqslant 0}\Big[a_{k+1}\alpha^\vee_{p+1-k}\big(\eta_{p-k}(t)\big) + a_k \min_{t\geqslant s\geqslant 0}\alpha^\vee_{p-k}\big(\eta_{p-k}(s)\big)\Big],$$
$$W_k = a_k \min_{T\geqslant t\geqslant 0}\alpha^\vee_{p-k}\big(\eta_{p-k}(t)\big) - a_{k+1}\min_{T\geqslant t\geqslant 0}\alpha^\vee_{p+1-k}\big(\eta_{p+1-k}(t)\big).$$

Since $a_{k-1}+a_{k+1}=ca_k$, the lemma above gives that $V_k=\min(W_{k+1},V_{k+1})$. Therefore

$$V_0=\min(W_1,W_2,\ldots,W_p,V_p).$$

Notice that

$$V_p = \min_{T\geqslant t\geqslant 0}\Big[a_{p+1}\alpha^\vee_1\big(\eta_0(t)\big) + a_p \min_{t\geqslant s\geqslant 0}\alpha^\vee_0\big(\eta_0(s)\big)\Big]=0$$

and $W_p=a_{p+1}x_1$ since $\eta_0=\mathcal{P}_{w_0}\eta$ is dominant. Furthermore

$$V_0=\min_{0\leqslant t\leqslant T}\alpha^\vee_{p+1}\big(\eta_p(t)\big)$$

since $a_0=0$ and $a_1=1$. Hence,

$$\min_{0\leqslant t\leqslant T}\alpha^\vee_{p+1}\big(\eta_p(t)\big)=\min(a_2x_p-a_1x_{p-1},\ldots,a_px_2-a_{p-1}x_1,a_{p+1}x_1,0). \tag{4.2}$$

The path $\eta_{m-1}=\mathcal{P}_{\alpha_m}\eta$ is α_m-dominant, therefore $\alpha^\vee_m(\eta_{m-1}(t))\geqslant 0$ and it follows from (4.2) applied with $p=m-1$ that for $k=1,\ldots,m-2$

$$a_{m-k}x_{k+1}-a_{m-k-1}x_k\geqslant 0,$$

which is equivalent, since $a_{m-k}=a_k$ to

$$\frac{x_{m-1}}{a_{m-1}}\geqslant\frac{x_{m-2}}{a_{m-2}}\geqslant\cdots\geqslant\frac{x_1}{a_1}\geqslant 0.$$

Conversely, we suppose that these inequalities hold, i.e. that for $k=1,\ldots,m-2$

$$a_{k+1}x_{m-k}-a_kx_{m-k-1}\geqslant 0,$$
$$a_{m-k}x_{k+1}-a_{m-k-1}x_k\geqslant 0, \tag{4.3}$$

and that $(x_1,\ldots,x_m)\in K_\pi$ for some dominant path π. Let us show that

$$\eta=\mathcal{H}^{x_m}_{\alpha_m}\cdots\mathcal{H}^{x_1}_{\alpha_1}\pi$$

is well defined. Since the string parametrization of η is x this will prove the proposition. It is enough to show, by induction on $p=0,\ldots,m$ that

$$\eta_p:=\mathcal{H}^{x_p}_{\alpha_p}\mathcal{H}^{x_{p-1}}_{\alpha_{p-1}}\cdots\mathcal{H}^{x_1}_{\alpha_1}\pi$$

is α_{p+1}-dominant. This is clear for $p=0$ since $\eta_0=\pi$ is dominant. If we suppose that this is true until $p-1$ can apply (4.2) and write that

$$\min_{0\leqslant t\leqslant T}\alpha_{p+1}^{\vee}\big(\eta_p(t)\big)=\min(a_2x_p-a_1x_{p-1},\dots,a_px_2-a_{p-1}x_1,a_{p+1}x_1,0).$$

Since $c\leqslant 2$, it is easy to see that

$$\frac{a_{n-1}}{a_n}\geqslant\frac{a_{n-2}}{a_{n-1}}$$

for $n\leqslant m-1$. Therefore,

$$\frac{x_{k+1}}{x_k}\geqslant\frac{a_{m-k-1}}{a_{m-k}}\geqslant\frac{a_{p-k}}{a_{p-k+1}}$$

and $\alpha_{p+1}^{\vee}(\eta_p(t))\geqslant 0$ for all $0\leqslant t\leqslant T$. □

In the definition of V_k and W_k in the proof above, replace the sequence (a_k) by the sequence (a_{k+1}). We obtain the following formula.

Proposition 4.8. *If* $y_m=-\min_{T\geqslant t\geqslant 0}\alpha_{m-1}^{\vee}(\eta_m(t))$, *then*

$$y_m=\max\{0,a_{m-1}x_{m-1}-a_{m-2}x_m,a_{m-2}x_{m-2}-a_{m-3}x_{m-1},\dots,a_2x_2-a_1x_3,a_1x_1\}.$$

4.6. Remark on Gelfand–Tsetlin cones

In the Weyl group case, the continuous cone $C_{\mathbf{i}}$ appears in the description of toric degenerations (see Caldero [5], Alexeev and Brion [1]). The polytopes $C_{\mathbf{i}}^{\pi}$ are called the string polytopes in Alexeev and Brion [1]. Notice that they have shown that the classical Duistermaat–Heckman measure coincides with the one given below in Definition 5.4. Explicit inequalities for the string cone $C_{\mathbf{i}}$ (and therefore for the string polytopes) in the Weyl group case are given in full generality in Berenstein and Zelevinsky in [2, Thm. 3.12]. Before, Littelmann [24, Thm. 4.2] has described it for the so called “nice decompositions” of w_0. As explained in that paper they were introduced to generalize the Gelfand–Tsetlin cones.

For the convenience of the reader let us reproduce the description $C_{\mathbf{i}}$ in the A_n case, considered explicitly in Alexeev and Brion [1], for the standard reduced decomposition of the longest element in the symmetric group $W=S_{n+1}$. This decomposition $\mathbf{i}$ is

$$w_0=(s_1)(s_2s_1)(s_3s_2s_1)\dots(s_ns_{n-1}\dots s_1),$$

where s_i denotes the transposition exchanging i with $i+1$. Let us use on V the coordinates $x_{i,j}$ with $i,j\geqslant 1$, $i+j\leqslant n+1$. The string cone is defined by

$$x_{n,1}\geqslant 0;\quad x_{n-1,2}\geqslant x_{n-1,1}\geqslant 0;\quad \dots;\quad x_{1,n}\geqslant\dots\geqslant x_{1,1}\geqslant 0,$$

and to define the polyhedron $C_{\mathbf{i}}^{\pi}$ one has to add the inequalities

$$x_{i,j}\leqslant\alpha_j^{\vee}(\lambda)-x_{i,j-1}+\sum_{k=1}^{i-1}(-x_{k,j-1}+2x_{k,j}-x_{k,j+1})$$

where $\lambda = \pi(T)$. A more familiar description of this cone is in terms of Gelfand–Tsetlin patterns:

$$g_{i,j} \geqslant g_{i+1,j} \geqslant g_{i,j+1}$$

where $g_{0,j} = \lambda_j$ and $g_{i,j} = \lambda_j + \sum_{k=1}^{i}(x_{k,j-1} - x_{k,j})$ for $i, j \geqslant 1$, $i + j \leqslant n + 1$.

4.7. Crystal structure of the Littelmann module

We now return to the general case of a finite Coxeter group. Let π be a dominant path in $C_T^0(V)$. The geometry of the crystal L_π is easy to describe, using the sets $C_{\mathbf{i}}^\pi$ which parametrize L_π. We have seen (Theorem 4.5) that $C_{\mathbf{i}}^\pi$ depend on the path π only through $\pi(T)$. We put on $C_{\mathbf{i}}^\pi$ a continuous crystal structure in the following way. Let $\mathbf{i} = (s_1, \dots, s_q)$ where $w_0 = s_1 \dots s_q$ is a reduced decomposition. If $x = (x_1, \dots, x_q) \in C_{\mathbf{i}}^\pi$ we set

$$wt(x) = \pi(T) - \sum_{k=1}^{q} x_k \alpha_{s_k}.$$

If the simple root α is α_{s_1} then first define $e_{\alpha,\mathbf{i}}^r$ for $r \in \mathbb{R}$ by

$$e_{\alpha,\mathbf{i}}^r(x_1, x_2, \dots, x_q) = (x_1 + r, x_2, \dots, x_q) \text{ or } \mathbf{0}$$

depending whether $(x_1 + r, \dots, x_q)$ is in $C_{\mathbf{i}}^\pi$ or not. We let, for $b \in C_{\mathbf{i}}^\pi$,

$$\varepsilon_\alpha(b) = \max\{r \geqslant 0;\ e_{\alpha,\mathbf{i}}^r(b) \neq \mathbf{0}\}$$

and

$$\varphi_\alpha(b) = \max\{r \geqslant 0;\ e_{\alpha,\mathbf{i}}^{-r}(b) \neq \mathbf{0}\}.$$

We now consider the case where α is not α_1. We choose a reduced decomposition $w_0 = s_1' s_2' \dots s_q'$ with $\alpha_{s_1'} = \alpha$ and let $\mathbf{j} = (s_1', s_2', \dots, s_q')$. We can define $e_{\alpha,\mathbf{j}}^r$ on $C_{\mathbf{j}}^\pi$, ε_α, ϕ_α as above, and transport this action on $C_{\mathbf{i}}^\pi$ by the piecewise linear map $\phi_{\mathbf{i}}^{\mathbf{j}}$ introduced in Theorem 3.12. In other words,

$$e_{\alpha,\mathbf{i}}^r = \phi_{\mathbf{i}}^{\mathbf{j}} \circ e_{\alpha,\mathbf{j}}^r \circ \phi_{\mathbf{j}}^{\mathbf{i}}.$$

Finally we define the crystal operators by $e_\alpha^r = e_{\alpha,\mathbf{i}}^r$. Then $\rho_{\mathbf{i}} : L_\pi \to C_{\mathbf{i}}^\pi$ is an isomorphism of crystal. This first shows that our construction does not depend on the chosen decompositions $w_0 = s_1' s_2' \dots s_q'$ and then that the crystal structure on L_π depends only on the extremity $\pi(T)$ of the path π:

Theorem 4.9. *If π and $\bar{\pi}$ are two dominant paths such that $\pi(T) = \bar{\pi}(T)$ then the crystals on L_π and $L_{\bar{\pi}}$ are isomorphic.*

This is the analogue of Littelmann independence theorem (see [23]).

Definition 4.10. When W is finite, for $\lambda \in \overline{C}$, we denote $B(\lambda)$ the class of the continuous crystals isomorphic to L_π where π is a dominant path such that $\pi(T) = \lambda$.

4.8. Concatenation and closed crystals

The concatenation $\pi \star \eta$ of two paths $\pi : [0, T] \to V$, $\eta : [0, T] \to V$ is defined in Littelmann [23] as the path $\pi \star \eta : [0, T] \to V$ given by $(\pi \star \eta)(t) = \pi(2t)$, and $(\pi \star \eta)(t + T/2) = \pi(T) + \eta(2t)$ when $0 \leqslant t \leqslant T/2$. The following theorem is instrumental to prove uniqueness.

Theorem 4.11. *The map*

$$\Theta : C_T^0(V) \otimes C_T^0(V) \to C_T^0(V)$$

defined by $\Theta(\eta_1 \otimes \eta_2) = \eta_1 \star \eta_2$ *is a crystal isomorphism.*

Proof. We have to show that, for simple roots α, for $\eta_1 \in L_{\pi_1}$, $\eta_2 \in L_{\pi_2}$, for all $s \in \mathbb{R}$,

$$\Theta\big[e_\alpha^s(\eta_1 \otimes \eta_2)\big] = \mathcal{E}_\alpha^s(\eta_1 \star \eta_2).$$

This is a purely one-dimensional statement, which uses only one root, hence it follows from the similar fact for Littelmann and Kashiwara crystals. For the convenience of the reader we provide a proof. For any $x \geqslant 0$, let

$$\mathcal{P}_\alpha^x \eta(t) = \eta(t) - \min\Big(0, x + \inf_{0 \leqslant s \leqslant t} \alpha^\vee \eta(s)\Big)\alpha.$$

Thus, for $y = (-\inf_{0 \leqslant s \leqslant T} \alpha^\vee \eta(s) - x) \vee 0$,

$$\mathcal{P}_\alpha^x \eta = \mathcal{E}_\alpha^y \eta. \tag{4.4}$$

Lemma 4.12. *Let* $\eta_1, \eta_2 \in C_T^0(V)$, *then*

(i) $\mathcal{P}_\alpha(\eta_1 \star \eta_2) = \mathcal{P}_\alpha \eta_1 \star \mathcal{P}_\alpha^x \eta_2$ *where* $x = \alpha^\vee \eta_1(T) - \inf_{0 \leqslant t \leqslant T} \alpha^\vee \eta_1(t)$;
(ii) *if* $x \geqslant 0$, $\mathcal{P}_\alpha \mathcal{P}_\alpha^x = \mathcal{P}_\alpha$;
(iii) *if* $x \geqslant 0$, $y \in [0, \alpha^\vee \pi(T)]$, *and* π *be an* α*-dominant path,* $\mathcal{P}_\alpha^x \mathcal{H}_\alpha^y \pi = \mathcal{H}_\alpha^{x \wedge y} \pi$.

Proof. For all $t \in [0, T/2]$, $\mathcal{P}_\alpha(\eta_1 \star \eta_2)(t) = \mathcal{P}_\alpha \eta_1(t)$. Furthermore,

$$\begin{aligned}
&\mathcal{P}_\alpha(\eta_1 \star \eta_2)\big((T+t)/2\big) \\
&\quad = (\eta_1 \star \eta_2)\big((T+t)/2\big) - \min\Big[\inf_{0 \leqslant s \leqslant T} \alpha^\vee \eta_1(s), \alpha^\vee \eta_1(T) + \inf_{0 \leqslant s \leqslant t} \alpha^\vee \eta_2(s)\Big]\alpha \\
&\quad = \eta_1(T) - \inf_{0 \leqslant s \leqslant T} \alpha^\vee \eta_1(s)\alpha \\
&\qquad + \eta_2(t) - \min\Big[0, \inf_{0 \leqslant s \leqslant t} \alpha^\vee \eta_2(s) + \alpha^\vee \eta_1(T) - \inf_{0 \leqslant s \leqslant T} \alpha^\vee \eta_1(s)\Big]\alpha \\
&\quad = \mathcal{P}_\alpha \eta_1(T) + \mathcal{P}_\alpha^x \eta_2(t).
\end{aligned}$$

This proves (i), and (ii) follows from (4.4). Furthermore, $\inf_{0 \leqslant s \leqslant T} \alpha^\vee(\mathcal{H}_\alpha^y \pi(s)) = -y$, therefore (iii) follows also from (4.4). $\square$

Proposition 4.13. *Let π_1, π_2 be α-dominant paths, $x \in [0, \alpha^\vee \pi_1(T)]$, $y \in [0, \alpha^\vee \pi_2(T)]$, $z = \min(y, \alpha^\vee \pi_1(T) - x)$ and $r = x + y - z$, then*

$$\mathcal{H}_\alpha^x \pi_1 \star \mathcal{H}_\alpha^y \pi_2 = \mathcal{H}_\alpha^r \big(\pi_1 \star \mathcal{H}_\alpha^z \pi_2\big).$$

Proof. Let $s = \alpha^\vee(\mathcal{H}_\alpha^x \pi_1(T)) - \inf_{0 \leqslant t \leqslant T} \alpha^\vee(\mathcal{H}_\alpha^x \pi_1)(t)$. By Lemma 4.12:

$$\mathcal{P}_\alpha\big(\mathcal{H}_\alpha^x \pi_1 \star \mathcal{H}_\alpha^y \pi_2\big) = \mathcal{P}_\alpha\big(\mathcal{H}_\alpha^x \pi_1\big) \star \mathcal{P}_\alpha^s\big(\mathcal{H}_\alpha^y \pi_2\big)$$

and $\mathcal{P}_\alpha^s \mathcal{H}_\alpha^y \pi_2 = \mathcal{H}_\alpha^{s \wedge y} \pi_2$. Since $\mathcal{P}_\alpha \mathcal{H}_\alpha^x \pi_1 = \pi_1$ one has

$$\mathcal{P}_\alpha\big(\mathcal{H}_\alpha^x \pi_1 \star \mathcal{H}_\alpha^y \pi_2\big) = \pi_1 \star \mathcal{H}_\alpha^{s \wedge y} \pi_2.$$

Notice that $s = \alpha^\vee(\pi_1(T)) - x$. On the other hand,

$$\begin{aligned}
\big(\mathcal{H}_\alpha^x \pi_1 \star \mathcal{H}_\alpha^y \pi_2\big)(T) &= \mathcal{H}_\alpha^x \pi_1(T) + \mathcal{H}_\alpha^y \pi_2(T) = \pi_1(T) + \pi_2(T) - (x + y)\alpha, \\
\big(\pi_1 \star \mathcal{H}_\alpha^{s \wedge y} \pi_2\big)(T) &= \pi_1(T) + \pi_2(T) - (s \wedge y)\alpha
\end{aligned}$$

and we know that $\eta = \mathcal{H}_\alpha^r \pi$ is characterized by the properties $\mathcal{P}_\alpha \eta = \pi$ and $\eta(T) = \pi(T) - r\alpha$. Therefore the proposition holds for $r + s \wedge y = x + y$. □

We now prove that, for $\alpha \in \Sigma$, $\eta_1 \in L_{\pi_1}$, $\eta_2 \in L_{\pi_2}$, for all $s \in \mathbb{R}$,

$$\Theta\big[e_\alpha^s(\eta_1 \otimes \eta_2)\big] = \mathcal{E}_\alpha^s(\eta_1 \star \eta_2).$$

Since $e_\alpha^s e_\alpha^t = e_\alpha^{s+t}$ and $\mathcal{E}_\alpha^s \mathcal{E}_\alpha^t = \mathcal{E}_\alpha^{s+t}$ it is sufficient to check this for s near 0. We write $\eta_1 = \mathcal{H}_\alpha^x \pi_1$ and $\eta_2 = \mathcal{H}_\alpha^y \pi_2$ where $\pi_1 = \mathcal{P}_\alpha(\eta_1), \pi_2 = \mathcal{P}_\alpha(\eta_2)$ are α-dominant. By Proposition 4.13, if $z = \min(y, \alpha^\vee \pi_1(T) - x)$ and $r = x + y - z$, then

$$\mathcal{E}_\alpha^s(\eta_1 \star \eta_2) = \mathcal{E}_\alpha^s\big(\mathcal{H}_\alpha^x \pi_1 \star \mathcal{H}_\alpha^y \pi_2\big) = \mathcal{E}_\alpha^s \mathcal{H}_\alpha^r\big(\pi_1 \star \mathcal{H}_\alpha^z \pi_2\big).$$

We first show that if

$$\mathcal{E}_\alpha^s(\eta_1 \star \eta_2) = \mathbf{0} \tag{4.5}$$

then $e_\alpha^s(\eta_1 \otimes \eta_2) = \mathbf{0}$. For $|s|$ small enough (4.5) holds only when $r = 0$ and $s > 0$ or when $s < 0$ and

$$r = \alpha^\vee\big(\big(\pi_1 \star \mathcal{H}_\alpha^z \pi_2\big)(T)\big) = \alpha^\vee \pi_1(T) + \alpha^\vee \pi_2(T) - 2z. \tag{4.6}$$

If $r = 0$, then $z = \min(y, \alpha^\vee \pi_1(T) - x) = x + y$ hence $x = 0$ and $y \leqslant \alpha^\vee \pi_1(T)$. But

$$\varepsilon_\alpha(\eta_1 \otimes \eta_2) = \varepsilon_\alpha(\eta_1) - \min\big(\varphi_\alpha(\eta_1) - \varepsilon_\alpha(\eta_2), 0\big) = \max\big(2x + y - \alpha^\vee \pi_1(T), x\big)$$

(notice that, in general, when π is α-dominant, $\varepsilon_\alpha(\mathcal{H}_\alpha^x \pi) = x$ and $\varphi_\alpha(\mathcal{H}_\alpha^x \pi) = \alpha^\vee \pi(T) - x$). Therefore $\varepsilon_\alpha(\eta_1 \otimes \eta_2) = 0$ and $e_\alpha^s(\eta_1 \otimes \eta_2) = \mathbf{0}$. Now, if r is given by (4.6), then

$$z = \alpha^\vee \pi_1(T) - x + \alpha^\vee \pi_2(T) - y$$

since $r = x + y - z$. We know that $\alpha^\vee \pi_2(T) - y \geqslant 0$, hence $z = \min(y, \alpha^\vee \pi_1(T) - x)$ only if

$$z = \alpha^\vee \pi_1(T) - x, \qquad \alpha^\vee \pi_2(T) = y, \qquad y \geqslant \alpha^\vee \pi_1(T) - x.$$

Then

$$\varepsilon_\alpha(\eta_1 \otimes \eta_2) = 2x + y - \alpha^\vee \pi_1(T).$$

On the other hand,

$$wt(\eta_1 \otimes \eta_2) = wt(\eta_1) + wt(\eta_2) = \pi_1(T) - x\alpha + \pi_2(T) - y\alpha,$$

thus, using $y = \alpha^\vee \pi_2(T)$,

$$\varphi_\alpha(\eta_1 \otimes \eta_2) = \varepsilon_\alpha(\eta_1 \otimes \eta_2) + \alpha^\vee\big(wt(\eta_1 \otimes \eta_2)\big) = 0$$

and $e_\alpha^s(\eta_1 \otimes \eta_2) = \mathbf{0}$ when $s < 0$.

We now consider the case where (4.5) does not hold. Then for s small enough,

$$\mathcal{E}_\alpha^s(\eta_1 \star \eta_2) = \mathcal{E}_\alpha^s \mathcal{H}_\alpha^r\big(\pi_1 \star \mathcal{H}_\alpha^z \pi_2\big) = \mathcal{H}_\alpha^{r-s}\big(\pi_1 \star \mathcal{H}_\alpha^z \pi_2\big).$$

Using Proposition 4.13, if s is small enough, and $y > \alpha^\vee \pi_1(T) - x$, then

$$\mathcal{H}_\alpha^{r-s}\big(\pi_1 \star \mathcal{H}_\alpha^z \pi_2\big) = \mathcal{H}_\alpha^{x-s} \pi_1 \star \mathcal{H}_\alpha^y \pi_2 = \Theta\big(e_\alpha^s\big(\mathcal{H}_\alpha^x \pi_1 \otimes \mathcal{H}_\alpha^y \pi_2\big)\big)$$

and if $y < \alpha^\vee \pi_1(T) - x$, then

$$\mathcal{H}_\alpha^{r-s}\big(\pi_1 \star \mathcal{H}_\alpha^z \pi_2\big) = \mathcal{H}_\alpha^x \pi_1 \star \mathcal{H}_\alpha^{y-s} \pi_2 = \Theta\big(e_\alpha^s\big(\mathcal{H}_\alpha^x \pi_1 \otimes \mathcal{H}_\alpha^y \pi_2\big)\big).$$

The end of the proof is straightforward. □

By Theorem 4.9, this proves that the family of crystals $B(\lambda)$, $\lambda \in \overline{C}$ is closed. From Theorems 3.11 and 2.6, we get

Theorem 4.14. *When W is a finite Coxeter group, there exists one and only one closed family of highest weight normal continuous crystals $B(\lambda)$, $\lambda \in \overline{C}$.*

4.9. Action of W on the Littelmann crystal

Following Kashiwara [20,21] and Littelmann [23], we show that we can define an action of the Coxeter group on each crystal L_π. We first notice that for each simple root α, we can define an involution S_α on the set of paths by

$$S_\alpha \eta = \mathcal{E}_\alpha^x \eta \quad \text{for } x = -\alpha^\vee\big(\eta(T)\big).$$

In particular,

$$S_\alpha \eta(T) = s_\alpha\big(\eta(T)\big). \tag{4.7}$$

Lemma 4.15. *Let $\eta \in C_T^0(V)$ and $\alpha \in \Sigma$ such that $\alpha^\vee(\eta(T)) < 0$. For each $\gamma \in C_T^0(V)$ there exists $m \in \mathbb{N}$ such that, for all $n \geqslant 0$,*

$$\mathcal{P}_\alpha\big(\gamma \star \eta^{\star(m+n)}\big) = \mathcal{P}_\alpha\big(\gamma \star \eta^{\star m}\big) \star S_\alpha(\eta)^{\star n}.$$

Proof. By Lemma 4.12,

$$\mathcal{P}_\alpha\big(\gamma \star \eta^{\star(n+1)}\big) = \mathcal{P}_\alpha\big(\gamma \star \eta^{\star n}\big) \star \mathcal{P}_\alpha^x(\eta)$$

where

$$x = \alpha^\vee\big(\gamma \star \eta^{\star n}\big)(T) - \min_{0 \leqslant s \leqslant T} \alpha^\vee\big(\gamma \star \eta^{\star n}\big)(s).$$

Let $\gamma_{\min} = \min_{0 \leqslant s \leqslant T} \alpha^\vee \gamma(s)$ and $\eta_{\min} = \min_{0 \leqslant s \leqslant T} \alpha^\vee \eta(s)$. Since $\alpha^\vee \gamma(T) < 0$, there exists $m > 0$ such that for $n \geqslant m$ one has,

$$\begin{aligned}\min_{0 \leqslant s \leqslant T} \alpha^\vee\big(\gamma \star \eta^{\star n}\big)(s) &= \min\big(\gamma_{\min}, \alpha^\vee\big(\gamma(T) + k\eta(T)\big) + \eta_{\min};\ 0 \leqslant k \leqslant n-1\big) \\ &= \alpha^\vee\big(\gamma(T) + (n-1)\eta(T)\big) + \eta_{\min}.\end{aligned}$$

Using that $(\gamma \star \eta^{\star m})(T) = \gamma(T) + m\eta(T)$ we have $x = \alpha^\vee \eta(T) - \eta_{\min}$. In this case, $\mathcal{P}_\alpha^x(\eta) = S_\alpha(\eta)$, which proves the lemma by induction on $n \geqslant m$. $\square$

Theorem 4.16. *There is an action $\{S_w,\ w \in W\}$ of the Coxeter group W on each L_π such that $S_{s_\alpha} = S_\alpha$ when α is a simple root.*

Proof. By Matsumoto's lemma, it suffices to prove that the transformations S_α satisfy to the braid relations. Therefore we can assume that W is a dihedral group $I(q)$. Consider two roots α, β generating W. Let η be a path, there exists a sequence $(\alpha_i) = \alpha, \beta, \alpha, \ldots$ or $\beta, \alpha, \beta, \ldots$ such that $s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_r} \eta(T) \in -\overline{C}$. Let $\tilde{\eta} = S_{\alpha_1} S_{\alpha_2} \ldots S_{\alpha_r} \eta$. Let $s_{\alpha_q} \ldots s_{\alpha_1}$ be a reduced decomposition. We show by induction on $k \leqslant q$ that there exists $m_k \geqslant 0$ and a path γ_k such that

$$\mathcal{P}_{\alpha_k} \ldots \mathcal{P}_{\alpha_1}\big(\tilde{\eta}^{\star(m_k+n)}\big) = \gamma_k \star (S_{\alpha_k} \ldots S_{\alpha_1} \tilde{\eta})^{\star n}. \tag{4.8}$$

For $k = 1$, this is the preceding lemma. Suppose that this holds for some k. Then

$$\alpha_{k+1}^\vee\big(S_{\alpha_k} \ldots S_{\alpha_1} \tilde{\eta}(T)\big) \leqslant 0$$

(cf. Bourbaki [4, Ch. 5, No. 4, Thm. 1]). Thus, by the lemma, there exists m such that, for $n \geqslant 0$,

$$\mathcal{P}_{\alpha_{k+1}}\big(\gamma_k \star (S_{\alpha_k} \ldots S_{\alpha_1} \tilde{\eta})^{\star(m+n)}\big) = \mathcal{P}_{\alpha_{k+1}}\big(\gamma_k \star (S_{\alpha_k} \ldots S_{\alpha_1} \tilde{\eta})^{\star m}\big) \star (S_{\alpha_{k+1}} S_{\alpha_k} \ldots S_{\alpha_1} \tilde{\eta})^{\star n}.$$

Hence, by the induction hypothesis, if $\gamma_{k+1} = \mathcal{P}_{\alpha_{k+1}}(\gamma_k \star (S_{\alpha_k} \ldots S_{\alpha_1} \tilde{\eta})^{\star m})$, then

$$\mathcal{P}_{\alpha_{k+1}} \mathcal{P}_{\alpha_k} \ldots \mathcal{P}_{\alpha_1}\big(\tilde{\eta}^{\star(m_k+m+n)}\big) = \gamma_{k+1} \star (S_{\alpha_{k+1}} S_{\alpha_k} \ldots S_{\alpha_1} \tilde{\eta})^{\star n}.$$

We apply (4.8) with $k = q$, then there exists two reduced decompositions, and we see that $S_{\alpha_q} S_{\alpha_{q-1}} \ldots S_{\alpha_1} \tilde{\eta}$ does not depend on the reduced decomposition because the left-hand side does not, by the braid relations for the $\mathcal{P}_\alpha$. This implies easily that $S_{\alpha_q} S_{\alpha_{q-1}} \dots S_{\alpha_1}\eta$ also does not depend on the reduced decomposition. □

Using the crystal isomorphism between L_π and the crystal $B(\pi(T))$ we see that

Corollary 4.17. *The Coxeter group W acts on each crystal $B(\lambda)$, where $\lambda \in \overline{C}$, in such a way that, for $s = s_\alpha$ in S, and $b \in B(\lambda)$,*

$$S_\alpha(b) = e_\alpha^x(b), \quad \textit{where } x = -\alpha^\vee\big(wt(b)\big).$$

Notice that these S_α are not crystal morphisms.

4.10. Schützenberger involution

The classical Schützenberger involution associates to a Young tableau T another Young tableau $\hat{T}$ of the same shape. If (P, Q) is the pair associated by Robinson–Schensted–Knuth (RSK) algorithm to the word $u_1 \dots u_n$ in the letters $1, \dots, k$, then $(\hat{P}, \hat{Q})$ is the pair associated with $u_n^* \dots u_1^*$ where $i^* = k + 1 - i$, see, e.g., Fulton [9]. It is remarkable that $\hat{P}$ depends only on P, and that $\hat{Q}$ depends only on Q. We will establish an analogous property for the analogue of the Schützenberger involution defined in [3] for finite Coxeter groups. The crystallographic case has been recently investigated by Henriques and Kamnitzer [13,14], and Morier-Genoud [27].

For any path $\eta \in C_T^0(V)$, let $\kappa\eta(t) = \eta(T - t) - \eta(T)$, $0 \leqslant t \leqslant T$, and

$$S\eta = -w_0\kappa\eta.$$

Since $w_0^2 = id$, S is an involution of $C_T^0(V)$. The following is proved in [3].

Proposition 4.18. *For any $\eta \in C_T^0(V)$, $\mathcal{P}_{w_0} S\eta(T) = \mathcal{P}_{w_0}\eta(T)$.*

As remarked in [3], this implies that the transformation on dominant paths

$$\pi \mapsto I\pi = \mathcal{P}_{w_0} S\pi$$

gives the analogue of the Schützenberger involution on the Q's. We will consider the action on the crystal itself, i.e. the analogue of the Schützenberger involution on the P's. For each dominant path $\pi \in C_T^0(V)$ the crystals L_π and $L_{I\pi}$ are isomorphic, since $\pi(T) = I\pi(T)$. Therefore there is a unique isomorphism $J_\pi : L_\pi \to L_{I\pi}$, it satisfies $J_\pi(\pi) = I\pi$. For each path $\eta \in C_T^0(V)$, let $J(\eta) = J_\pi(\eta)$, where $\pi = \mathcal{P}_{w_0}\eta$. This defines an involutive isomorphism of crystal $J : C_T^0(V) \to C_T^0(V)$. We will see that

$$\tilde{S} = J \circ S$$

is the analogue of the Schützenberger involution on crystals. Although $\tilde{S}$ is not a crystal isomorphism, and contrary to S, it conserves the crystal connected components since $\tilde{S}(L_\pi) = L_\pi$, for each dominant path π, this is the main reason for introducing it.

If α is a simple root, then $\tilde{\alpha} = -w_0\alpha$ is also a simple root and $\tilde{\alpha}^\vee = -\alpha^\vee w_0$. The following property is straightforward. In the A_n case, it was shown by Lascoux, Leclerc and Thibon [22] and Henriques and Kamnitzer [13] that it characterizes the Schützenberger involution.

Lemma 4.19. *For any path η in $C_T^0(V)$, any $r \in \mathbb{R}$, and any simple root α, one has*

$$\mathcal{E}_\alpha^r \tilde{S}\eta = \tilde{S}\mathcal{E}_{\tilde{\alpha}}^{-r}\eta,$$

$$\varepsilon_{\tilde{\alpha}}(\tilde{S}\eta) = \varphi_\alpha(\eta), \qquad \varphi_{\tilde{\alpha}}(\tilde{S}\eta) = \varepsilon_\alpha(\eta),$$

$$\tilde{S}\eta(T) = w_0\eta(T).$$

An important consequence of this lemma is that $\tilde{S}: L_\pi \to L_\pi$ depends only on the crystal structure of L_π. Indeed, if $\eta = \mathcal{E}_{\alpha_1}^{r_1} \cdots \mathcal{E}_{\alpha_k}^{r_k}\pi$ then $\tilde{S}(\eta) = \mathcal{E}_{\tilde{\alpha}_1}^{-r_1} \cdots \mathcal{E}_{\tilde{\alpha}_k}^{-r_k}\tilde{S}(\pi)$ and $\tilde{S}(\pi)$ is the unique element of L_π which has the lowest weight $w_0\pi(T)$, namely $S_{w_0}\pi$, where S_{w_0} is given by Theorem 4.16. In particular, using the isomorphism between L_π and $B(\lambda)$ where $\lambda = \pi(T)$, we can transport the action of $\tilde{S}$ on each $B(\lambda)$, $\lambda \in \overline{C}$.

Notice that $S \circ J$ also satisfies to this lemma. Therefore, by uniqueness,

$$S \circ J = J \circ S$$

thus $\tilde{S}$ is an involution. Following Henriques and Kamnitzer [14], let us show:

Theorem 4.20. *The map $\tau : C_T^0(V) \to C_T^0(V)$ defined by*

$$\tau(\eta_1 \star \eta_2) = \tilde{S}(\tilde{S}\eta_2 \star \tilde{S}\eta_1)$$

is an involutive crystal isomorphism.

Proof. Remark first that any path can be written uniquely as the concatenation of two paths, hence τ is well defined, furthermore $S(\eta_1 \star \eta_2) = S(\eta_2) \star S(\eta_1)$, therefore, since $\tilde{S} = SJ = JS$, and S is involutive,

$$\tau(\eta_1 \star \eta_2) = JS(SJ\eta_2 \star SJ\eta_1) = JS^2(J\eta_1 \star J\eta_2) = J(J\eta_1 \star J\eta_2).$$

Consider the map $J^{(2)} : C_T^0(V) \to C_T^0(V)$ defined by $J^{(2)}(\eta_1 \star \eta_2) = J\eta_1 \star J\eta_2$. Remark that $J^{(2)} = \Theta \circ (J \otimes J) \circ \Theta^{-1}$ where $\Theta : C_T^0(V) \otimes C_T^0(V) \to C_T^0(V)$ is the crystal isomorphism defined in Theorem 4.11 and $(J \otimes J)(\eta_1 \otimes \eta_2) = J(\eta_1) \otimes J(\eta_2)$. Since J is an isomorphism, this implies that $J^{(2)}$ is an isomorphism, thus $\tau = J \circ J^{(2)}$ is an isomorphism.

Let $\tilde{S}^{(2)}$ be defined by $\tilde{S}^{(2)}(\eta_1 \star \eta_2) = \tilde{S}(\eta_2) \star \tilde{S}(\eta_1)$. Then $\tau = \tilde{S} \circ \tilde{S}^{(2)}$, and, since $\tilde{S}$ is an involution, the inverse of τ is $\tilde{S}^{(2)} \circ \tilde{S}$. So to prove that τ is an involution we have to show that $\tilde{S} \circ \tilde{S}^{(2)} = \tilde{S}^{(2)} \circ \tilde{S}$. Both these maps are crystal isomorphisms, so it is enough to check that for any $\eta \in C_T^0(V)$, the two paths $(\tilde{S} \circ \tilde{S}^{(2)})(\eta)$ and $(\tilde{S}^{(2)} \circ \tilde{S})(\eta)$ are in the same connected crystal component. Since $\tilde{S}$ conserves each connected component, η and $\tilde{S}(\eta)$ on the one hand, and $\tilde{S}^{(2)}(\eta)$ and $\tilde{S}(\tilde{S}^{(2)}(\eta))$ on the other hand, are in the same component. Therefore is it sufficient to show that if η and μ are in the same component then $\tilde{S}^{(2)}(\eta)$ and $\tilde{S}^{(2)}(\mu)$ are in the same component. Let us write $\eta = \eta_1 \star \eta_2$. Then if $\mu = \mathcal{E}_\alpha^r(\eta)$, $\sigma = \varphi_\alpha(\eta_1) - \varepsilon_\alpha(\eta_2)$ and $\tilde{\sigma} = -\sigma$,

$$\tilde{S}\big(\mathcal{E}_\alpha^{\min(r,-\sigma)+\sigma^+}\eta_2\big) = \mathcal{E}_{\tilde{\alpha}}^{-\min(r,-\sigma)-\sigma^+}\tilde{S}\eta_2 = \mathcal{E}_{\tilde{\alpha}}^{\max(-r,\tilde{\sigma})-\tilde{\sigma}^-}\tilde{S}\eta_2$$

and

$$\tilde{S}\big(\mathcal{E}_{\alpha}^{\max(r,-\sigma)-\sigma^-}\eta_1\big) = \mathcal{E}_{\tilde{\alpha}}^{-\max(r,-\sigma)+\sigma^-}\tilde{\eta}_1 = \mathcal{E}_{\tilde{\alpha}}^{\min(-r,-\tilde{\sigma})+\tilde{\sigma}^+}\tilde{S}\eta_1,$$

therefore

$$\begin{aligned}
\tilde{S}^{(2)}(\mu) &= \tilde{S}^{(2)}\big(\mathcal{E}_{\alpha}^{\max(r,-\sigma)-\sigma^-}\eta_1 \star \mathcal{E}_{\alpha}^{\min(r,-\sigma)+\sigma^+}\eta_2\big) \\
&= \tilde{S}\big(\mathcal{E}_{\alpha}^{\min(r,-\sigma)+\sigma^+}\eta_2\big) \star \tilde{S}\big(\mathcal{E}_{\alpha}^{\max(r,-\sigma)-\sigma^-}\eta_1\big) \\
&= \mathcal{E}_{\tilde{\alpha}}^{\max(-r,\tilde{\sigma})-\tilde{\sigma}^-}\tilde{S}\eta_2 \star \mathcal{E}_{\tilde{\alpha}}^{\min(-r,-\tilde{\sigma})+\tilde{\sigma}^+}\tilde{S}\eta_1 \\
&= \mathcal{E}_{\tilde{\alpha}}^{-r}(\tilde{S}\eta_2 \star \tilde{S}\eta_1) \\
&= \mathcal{E}_{\tilde{\alpha}}^{-r}\tilde{S}^{(2)}(\eta).
\end{aligned}$$

So in this case $\tilde{S}^{(2)}(\mu)$ and $\tilde{S}^{(2)}(\eta)$ are in the same component. One concludes easily by induction. □

We can now define an involution $\tilde{S}_\lambda$ on each continuous crystal of the family $\{B(\lambda), \lambda \in \overline{C}\}$ by transporting the action of $\tilde{S}$ on $C_T^0(V)$. Let $\lambda, \mu \in \overline{C}$. For $b_1 \in B(\lambda)$ and $b_2 \in B(\mu)$ let

$$\tau_{\lambda,\mu}(b_1 \otimes b_2) = \tilde{S}_\gamma(\tilde{S}_\mu b_2 \otimes \tilde{S}_\lambda b_1)$$

where $\gamma \in \overline{C}$ is such that $\tilde{S}_\mu b_2 \otimes \tilde{S}_\lambda b_1 \in B(\gamma)$.

Theorem 4.21. *For $\lambda, \mu \in \overline{C}$, the map*

$$\tau_{\lambda,\mu} : B(\lambda) \otimes B(\mu) \to B(\mu) \otimes B(\lambda)$$

is a crystal isomorphism.

This follows from Theorem 4.20. As in the construction of Henriques and Kamnitzer [13,14] these isomorphisms do not obey the axioms for a braided monoidal category, but instead we have that:

(1) $\tau_{\mu,\lambda} \circ \tau_{\lambda,\mu} = 1$;
(2) the following diagram commutes:

$$\begin{array}{ccc}
B(\lambda) \otimes B(\mu) \otimes B(\sigma) & \xrightarrow{1\otimes\tau_{(\mu,\sigma)}} & B(\lambda) \otimes B(\sigma) \otimes B(\mu) \\
\Big\downarrow {\scriptstyle \tau_{(\lambda,\mu)}\otimes 1} & & \Big\downarrow {\scriptstyle \tau_{(\lambda,(\sigma,\mu))}} \\
B(\mu) \otimes B(\lambda) \otimes B(\sigma) & \xrightarrow[\tau_{((\mu,\lambda),\sigma)}]{} & B(\sigma) \otimes B(\mu) \otimes B(\lambda)
\end{array}$$

which makes of $B(\lambda)$, $\lambda \in \overline{C}$, a coboundary category.

5. The Duistermaat–Heckman measure and Brownian motion

5.1. In this section, we consider a finite Coxeter group, with a realization in some Euclidean space V identified with its dual so that, for each root α, $\alpha^\vee = \frac{2\alpha}{\|\alpha\|^2}$. We will introduce an analogue, for continuous crystals, of the Duistermaat–Heckman measure (see [7]), compute its Laplace transform (the analogue of the Harish-Chandra formula), and study its connections with Brownian motion.

5.2. Brownian motion and the Pitman transform

Fix a reduced decomposition of the longest word

$$w_0 = s_1 s_2 \dots s_q$$

and let $\mathbf{i} = (s_1, \dots, s_q)$. Recall that for any $\eta \in C_T^0(V)$, its string parameters $x = (x_1, \dots, x_q) = \varrho_{\mathbf{i}}(\eta)$ satisfy

$$0 \leqslant x_i \leqslant \alpha_{s_i}^\vee \left(\lambda - \sum_{j=1}^{i-1} x_j \alpha_{s_j} \right), \quad i \leqslant q, \tag{5.1}$$

where $\lambda = \mathcal{P}_{w_0}\eta(T)$. For each simple root α choose a reduced decomposition $\mathbf{i}_\alpha = (s_1^\alpha, \dots, s_q^\alpha)$ such that $s_1^\alpha = s_\alpha$ and denote the corresponding string parameters $\varrho_{\mathbf{i}_\alpha}(\eta)$ by $(x_1^\alpha, \dots, x_q^\alpha)$. Using the map $\phi_{\mathbf{i}}^{\mathbf{i}_\alpha}$ given by Theorem 3.12 we obtain a continuous piecewise linear function $\Psi_\alpha^{\mathbf{i}} : \mathbb{R}^q \to \mathbb{R}$ such that

$$x_1^\alpha = \Psi_\alpha^{\mathbf{i}}(x). \tag{5.2}$$

Of course

$$\Psi_\alpha^{\mathbf{i}}(x) \geqslant 0, \quad \text{for all } \alpha \in \Sigma. \tag{5.3}$$

Denote by $M_{\mathbf{i}}$ the set of $(x, \lambda) \in \mathbb{R}_+^q \times C$ which satisfy the inequalities (5.1) and (5.3), and set

$$M_{\mathbf{i}}^\lambda = \left\{ x \in \mathbb{R}_+^q \colon (x, \lambda) \in M_{\mathbf{i}} \right\}. \tag{5.4}$$

Let $\mathbb{P}$ be a probability measure on $C_T^0(V)$ under which η is a standard Brownian motion in V. We recall the following theorem from [3].

Theorem 5.1. *The stochastic process $\mathcal{P}_{w_0}\eta$ is a Brownian motion in V conditioned, in Doob's sense, to stay in the Weyl chamber $\overline{C}$.*

This means that $\mathcal{P}_{w_0}\eta$ is the h-process of the standard Brownian motion in V killed when it exits $\overline{C}$, for the harmonic function

$$h(\lambda) = \prod_{\alpha \in R_+} \alpha^\vee(\lambda),$$

for $\lambda \in V$, where R_+ is the set of all positive roots. Let $c_t = t^{q/2} \int_V e^{-\|\lambda\|^2/2t}\, d\lambda$ and

$$k = c_1^{-1} \int_C h(\lambda)^2 e^{-\|\lambda\|^2/2}\, d\lambda.$$

Theorem 5.2. *For $(\sigma, \lambda) \in M_{\mathbf{i}}$,*

$$\mathbb{P}\big(\varrho_{\mathbf{i}}(\eta) \in d\sigma,\ \mathcal{P}_{w_0}\eta(T) \in d\lambda\big) = c_T^{-1} h(\lambda) e^{-\|\lambda\|^2/2T}\, d\sigma\, d\lambda. \tag{5.5}$$

The conditional law of $\varrho_{\mathbf{i}}(\eta)$, given $(\mathcal{P}_{w_0}\eta(s),\ s \leqslant T)$ and $\mathcal{P}_{w_0}\eta(T) = \lambda$, is the normalized Lebesgue measure on $M_{\mathbf{i}}^{\lambda}$, and the volume of $M_{\mathbf{i}}^{\lambda}$ is $k^{-1}h(\lambda)$.

This theorem has the following interesting corollary, which gives a new proof of the fact that the set $C_{\mathbf{i}}^{\pi}$ depends only on $\pi(T)$, and is polyhedral.

Corollary 5.3. *For any dominant path π, let $\lambda = \pi(T)$, then $C_{\mathbf{i}}^{\pi} = M_{\mathbf{i}}^{\lambda}$, and*

$$C_{\mathbf{i}} = \big\{x \in \mathbb{R}_+^q;\ \Psi_{\alpha}^{\mathbf{i}}(x) \geqslant 0,\ \textit{for all } \alpha \in \Sigma\big\}.$$

Proof. It is clear that $C_{\mathbf{i}}^{\pi}$ is contained in $M_{\mathbf{i}}^{\lambda}$ and the theorem implies that $C_{\mathbf{i}}^{\pi}$, equal by definition to the set of $\varrho_{\mathbf{i}}(\eta)$ when $\mathcal{P}_{w_0}\eta = \pi$, contains $M_{\mathbf{i}}^{\lambda}$. The description of $C_{\mathbf{i}}$ follows, since $C_{\mathbf{i}} = \bigcup\{C_{\mathbf{i}}^{\pi}, \pi \text{ dominant path}\}$. □

Theorem 5.2 is proved in Section 5.4.

5.3. The Duistermaat–Heckman measure

Let G be a compact semi-simple Lie group with maximal torus T. If $\mathcal{O}_{\lambda}$ is a coadjoint orbit of G, corresponding to a dominant regular weight, endowed with its canonical symplectic structure ω, then this maximal torus acts on the symplectic manifold $(\mathcal{O}_{\lambda}, \omega)$, and the image of the Liouville measure on $\mathcal{O}_{\lambda}$ by the moment map, which takes values in the dual of the Lie algebra of T, is called the Duistermaat–Heckman measure. It is proved in [1] that this measure is the image of the Lebesgue measure on the Berenstein–Zelevinsky polytope by an affine map. In analogy with this case, we define for a realization of a finite Coxeter group, the Duistermaat–Heckman measure, and prove some properties which generalize the case of crystallographic groups.

Definition 5.4. For any $\lambda \in C$, the Duistermaat–Heckman measure $m_{\mathrm{DH}}^{\lambda}$ on V is the image of the Lebesgue measure on $M_{\mathbf{i}}^{\lambda}$ (defined by (5.4)) by the map

$$x = (x_1, \ldots, x_q) \in M_{\mathbf{i}}^{\lambda} \mapsto \lambda - \sum_{j=1}^{q} x_j \alpha_j \in V. \tag{5.6}$$

In the following, V^* denotes the complexification of V.

Theorem 5.5. *The Laplace transform of the Duistermaat–Heckman measure is given, for $z \in V^*$, by*

$$\int_V e^{\langle z,v\rangle} m_{\mathrm{DH}}^{\lambda}(dv) = \frac{\sum_{w\in W} \varepsilon(w) e^{\langle z, w\lambda\rangle}}{h(z)}, \tag{5.7}$$

where $\varepsilon(w)$ is the signature of $w \in W$.

With the notations of Theorem 5.2, the conditional law of $\eta(T)$, given $(\mathcal{P}_{w_0}\eta(s), 0 \leqslant s \leqslant T)$ and $\mathcal{P}_{w_0}\eta(T) = \lambda$, is the probability measure $\mu_{\mathrm{DH}}^{\lambda} = k m_{\mathrm{DH}}^{\lambda}/h(\lambda)$.

Formula (5.7) is the analogue, in our setting of the famous formula of Harish-Chandra [11]. Theorem 5.5 is proved in Section 5.5.

Proposition 5.6. *The Duistermaat–Heckman measure $m_{\mathrm{DH}}^{\lambda}$ has a continuous piecewise polynomial density, invariant under W and with support equal to the convex hull* $\mathrm{co}(W\lambda)$ *of $W\lambda$.*

Proof. The measure $m_{\mathrm{DH}}^{\lambda}$ is the image by an affine map of the Lebesgue measure on the convex polytope $C_{\mathbf{i}}^{\pi}$ when $\pi(T) = \lambda$. Therefore it has a piecewise polynomial density and a convex support. Its Laplace transform is invariant under W so $m_{\mathrm{DH}}^{\lambda}$ itself is invariant under W. The support $S(\lambda)$ of $m_{\mathrm{DH}}^{\lambda}/h(\lambda)$ is equal to $\{\eta(T);\ \eta \in L_\pi\}$. Notice that if η is in L_π, then when $x = \alpha^\vee(\eta(T))$, $\mathcal{E}_\alpha^x \eta$ is in L_π and $\mathcal{E}_\alpha^x \eta(T) = s_\alpha \eta(T)$. Starting from $\pi(T) = \lambda$ we thus see that $W\lambda$ is contained in $S(\lambda)$. So $\mathrm{co}(W\lambda)$ is contained in $S(\lambda)$. The components of $x \in M_{\mathbf{i}}^{\pi}$ are nonnegative, therefore $\mathrm{co}(W\lambda)$ contains $S(\lambda) \cap \overline{C}$ and, by W-invariance it contains $S(\lambda)$ itself. □

5.4. Proof of Theorem 5.2

First we recall some further path transformations which were introduced in [3]. For any positive root $\beta \in R_+$ (not necessarily simple), define $\mathcal{Q}_\beta = \mathcal{P}_\beta s_\beta$. Then, for $\psi \in C_T^0(V)$,

$$\mathcal{Q}_\beta \psi(t) = \psi(t) - \inf_{t \geqslant s \geqslant 0} \beta^\vee\big(\psi(t) - \psi(s)\big)\beta, \quad T \geqslant t \geqslant 0.$$

Let $w_0 = s_1 s_2 \dots s_q$ be a reduced decomposition, and let $\alpha_i = \alpha_{s_i}$. Since $s_\alpha \mathcal{P}_\beta = \mathcal{P}_{s_\alpha \beta} s_\alpha$, for roots $\alpha \neq \beta$, the following holds

$$\mathcal{Q}_{w_0} := \mathcal{P}_{w_0} w_0 = \mathcal{Q}_{\beta_1} \dots \mathcal{Q}_{\beta_q},$$

where $\beta_1 = \alpha_1$, $\beta_i = s_1 \dots s_{i-1}\alpha_i$, when $i \leqslant q$. Set $\psi_q = \psi$ and, for $i \leqslant q$,

$$\psi_{i-1} = \mathcal{Q}_{\beta_i} \dots \mathcal{Q}_{\beta_q} \psi \qquad y_i = -\inf_{T \geqslant t \geqslant 0} \beta_i^\vee\big(\psi_i(T) - \psi_i(t)\big). \tag{5.8}$$

Then $\psi_0 = \mathcal{Q}_{w_0}\psi$ and, for each $i \leqslant q$,

$$\mathcal{Q}_{w_0}\psi(T) = \psi_i(T) + \sum_{j=1}^{i} y_j \beta_j.$$

Define $\varsigma_{\mathbf{i}}(\psi) := (y_1, y_2, \ldots, y_q)$. Now let $\eta = w_0\psi$, so that $\mathcal{Q}_{w_0}\psi = \mathcal{P}_{w_0}\eta$. Set $\eta_q = \eta$ and, for $i \leqslant q$,

$$\eta_{i-1} = \mathcal{P}_{\alpha_i} \ldots \mathcal{P}_{\alpha_q}\eta, \qquad x_i = -\inf_{T \geqslant t \geqslant 0} \alpha_i^\vee\big(\eta_i(t)\big). \tag{5.9}$$

Then $\eta_0 = \mathcal{P}_{w_0}\eta$ and, for each $i \leqslant q$,

$$\mathcal{P}_{w_0}\eta(T) = \eta_i(T) + \sum_{j=1}^{i} x_j\alpha_j.$$

The parameters $\varrho_{\mathbf{i}}(\eta) = (x_1, \ldots, x_q)$ are related to $\varsigma_{\mathbf{i}}(\psi) = (y_1, y_2, \ldots, y_q)$ as follows.

Lemma 5.7. *For each $i \leqslant q$, we have:*

(i) $\eta_i = s_i \ldots s_1\psi_i$,
(ii) $x_i = y_i + \beta_i^\vee(\psi_i(T)) = \beta_i^\vee(\mathcal{Q}_{w_0}\psi(T) - \sum_{j=1}^{i-1} y_j\beta_j) - y_i$,
(iii) $y_i = x_i + \alpha_i^\vee(\eta_i(T)) = \alpha_i^\vee(\mathcal{P}_{w_0}\eta(T) - \sum_{j=1}^{i-1} x_j\alpha_j) - x_i$.

Proof. We prove (i) by induction on $i \leqslant q$. For $i = q$ it holds because $\eta_q = \eta = w_0\psi = w_0\psi_q$ and $s_q \ldots s_1 = w_0$. Note that, for each $i \leqslant q$, we can write

$$\mathcal{Q}_{\beta_i} = \mathcal{P}_{\beta_i} s_{\beta_i} = s_1 \ldots s_{i-1}\mathcal{P}_{\alpha_i} s_i \ldots s_1.$$

Therefore, assuming the induction hypothesis $\eta_i = s_i \ldots s_1\psi_i$,

$$\begin{aligned}\eta_{i-1} &= \mathcal{P}_{\alpha_i}\eta_i = \mathcal{P}_{\alpha_i} s_i \ldots s_1\psi_i \\ &= s_{i-1} \ldots s_1\mathcal{Q}_{\beta_i}\psi_i \\ &= s_{i-1} \ldots s_1\psi_{i-1},\end{aligned}$$

as required. This implies (ii), using $\eta_{i-1}(T) = \eta_i(T) + x_i\alpha_i$ and $\psi_{i-1}(T) = \psi_i(T) + y_i\beta_i$:

$$\begin{aligned}2x_i &= \alpha_i^\vee\big(\eta_{i-1}(T) - \eta_i(T)\big) \\ &= \alpha_i^\vee\big(s_{i-1} \ldots s_1\psi_{i-1}(T) - s_i \ldots s_1\psi_i(T)\big) \\ &= \alpha_i^\vee\big(s_{i-1} \ldots s_1\big(\psi_i(T) + y_i\beta_i\big) - s_i \ldots s_1\psi_i(T)\big) \\ &= 2y_i + \alpha_i^\vee\big(\alpha_i^\vee\big(s_{i-1} \ldots s_1\psi_i(T)\big)\alpha_i\big) \\ &= 2y_i + 2\beta_i^\vee\big(\psi_i(T)\big).\end{aligned}$$

Finally, (iii) follows immediately from (ii) and (i). □

This lemma shows that, when W is a Weyl group, then $(y_1, \ldots, y_q)$ are the Lusztig coordinates with respect to the decomposition $\mathbf{i}^*$ of the image of the path η with string coordinates $(x_1, \ldots, x_q)$ with respect to the decomposition $\mathbf{i}$ under the Schützenberger involution, where $\mathbf{i}^*$

is obtained from $\mathbf{i}$ by the map $\tilde{\alpha} = -w_0\alpha$ (see Morier-Genoud [27, Cor. 2.17]). By (iii) of the preceding lemma, we can define a mapping $F : M_{\mathbf{i}} \to \mathbb{R}_+^q \times C$ such that

$$\big(\varsigma_{\mathbf{i}}(\psi), \mathcal{Q}_{w_0}\psi(T)\big) = F\big(\varrho_{\mathbf{i}}(\eta), \mathcal{P}_{w_0}\eta(T)\big).$$

Let $L_{\mathbf{i}} = F(M_{\mathbf{i}})$. It follows from (ii) that $F^{-1}(y, \lambda) = (G(y, \lambda), \lambda)$, where

$$G(y, \lambda) = \beta_i^\vee\left(\lambda - \sum_{j=1}^{i-1} y_j\beta_j\right) - y_i.$$

Thus, $L_{\mathbf{i}}$ is the set of $(y, \lambda) \in \mathbb{R}_+^q \times C$ which satisfy

$$0 \leqslant y_i \leqslant \beta_i^\vee\left(\lambda - \sum_{j=1}^{i-1} y_j\beta_j\right) \quad (i \leqslant q) \tag{5.10}$$

and

$$\Psi_\alpha^{\mathbf{i}}\big(G(y, \lambda)\big) \geqslant 0, \quad \alpha \in \Sigma. \tag{5.11}$$

The analogue of Theorem 3.12 also holds for the parameters $\varsigma_{\mathbf{i}}(\psi) = (y_1, y_2, \ldots, y_q)$, and can be proved similarly. More precisely, for any two reduced decompositions $\mathbf{i}$ and $\mathbf{j}$, there is a piecewise linear map $\theta_{\mathbf{i}}^{\mathbf{j}} : \mathbb{R}^q \to \mathbb{R}^q$ such that $\varsigma_{\mathbf{j}}(\psi) = \theta_{\mathbf{i}}^{\mathbf{j}}(\varsigma_{\mathbf{i}}(\psi))$. In particular, for each simple root α, we can define a piecewise linear map $\Theta_\alpha^{\mathbf{i}} : \mathbb{R}^q \to \mathbb{R}$ such that, if $\mathbf{i}_\alpha = (s_1^\alpha, \ldots, s_q^\alpha)$ is a reduced decomposition with $s_1^\alpha = s_\alpha$, and $\varsigma_{\mathbf{i}_\alpha}(\psi) = (y_1^\alpha, y_2^\alpha, \ldots, y_q^\alpha)$, then $y_1^\alpha = \Theta_\alpha^{\mathbf{i}}(y)$ where $\varsigma_{\mathbf{i}}(\psi) = (y_1, y_2, \ldots, y_q)$. By Lemma 5.7, we have

$$\Theta_\alpha^{\mathbf{i}}(y) = \alpha^\vee(\lambda) - \Psi_\alpha^{\mathbf{i}}\big(G(y, \lambda)\big), \tag{5.12}$$

and the inequalities (5.11) can be written as

$$\alpha^\vee(\lambda) - \Theta_\alpha^{\mathbf{i}}(y) \geqslant 0, \quad \alpha \in \Sigma. \tag{5.13}$$

As in [3], we extend the definition of $\mathcal{Q}_\beta$ to two-sided paths. Denote by $C_{\mathbb{R}}^0(V)$ the set of continuous paths $\pi : \mathbb{R} \to V$ such that $\pi(0) = 0$ and $\alpha^\vee(\pi(t)) \to \pm\infty$ as $t \to \pm\infty$ for all simple α. For $\pi \in C_{\mathbb{R}}^0(V)$ and β a positive root, define $\mathcal{Q}_\beta\pi$ by

$$\mathcal{Q}_\beta\pi(t) = \pi(t) + \big[\omega(t) - \omega(0)\big]\beta,$$

where

$$\omega(t) = -\inf_{t \geqslant s > -\infty} \beta^\vee\big(\pi(t) - \pi(s)\big).$$

It is easy to see that $\mathcal{Q}_\beta\pi \in C_{\mathbb{R}}^0(V)$. Thus, we can set $\pi_q = \pi$ and, for $i \leqslant q$,

$$\pi_{i-1} = \mathcal{Q}_{\beta_i} \ldots \mathcal{Q}_{\beta_q}\pi, \qquad \omega_i(t) = -\inf_{s \leqslant t} \beta_i^\vee\big(\pi_i(t) - \pi_i(s)\big).$$

Then

$$\pi_0 = \mathcal{Q}_{w_0}\pi := \mathcal{Q}_{\beta_1}\dots\mathcal{Q}_{\beta_q}\pi$$

and, for each $i \leqslant q$,

$$\mathcal{Q}_{w_0}\pi(t) = \pi_i(t) + \sum_{j=1}^{i}\big[\omega_j(t) - \omega_j(0)\big]\beta_j.$$

For each $t \in \mathbb{R}$, write $\omega(t) = (\omega_1(t), \dots, \omega_q(t))$.

Lemma 5.8. *If $\mathcal{Q}_{w_0}\pi(t) = \lambda$ and $\omega(t) = y$ then*

$$\inf_{u\geqslant t} \alpha^\vee\big(\mathcal{Q}_{w_0}\pi(u)\big) = \alpha^\vee(\lambda) - \Theta_\alpha^{\mathbf{i}}(y).$$

Proof. It is straightforward to see that

$$\inf_{u\geqslant t} \beta_1^\vee\big(\mathcal{Q}_{w_0}\pi(u) - \mathcal{Q}_{w_0}\pi(t)\big) = \omega_1(t).$$

In particular, if $\mathbf{i}_\alpha = (s_1^\alpha, \dots, s_q^\alpha)$ is a reduced decomposition with $s_1^\alpha = s_\alpha$ and we denote the corresponding $\omega(\cdot)$ (defined as above) by $\omega^\alpha(\cdot)$, then

$$\inf_{u\geqslant t} \alpha^\vee\big(\mathcal{Q}_{w_0}\pi(u) - \mathcal{Q}_{w_0}\pi(t)\big) = \omega_1^\alpha(t).$$

Now let $\tau_0 = \tau_0^\alpha = t$ and, for $0 < i \leqslant q$,

$$\tau_i = \max\big\{s \leqslant \tau_{i-1}\colon \omega_i(s) = 0\big\}, \qquad \tau_i^\alpha = \max\big\{s \leqslant \tau_{i-1}^\alpha\colon \omega_i^\alpha(s) = 0\big\}.$$

Set $\tau = \min\{\tau_q, \tau_q^\alpha\}$. It is not hard to see that the path $\gamma \in C_{t-\tau}^0(V)$, defined by

$$\gamma(s) = \pi(\tau + s) - \pi(\tau), \quad t - \tau \geqslant s \geqslant 0,$$

satisfies $\varsigma_{\mathbf{i}}(\gamma) = \omega(t) = y$ and $\varsigma_{\mathbf{i}_\alpha}(\gamma) = \omega^\alpha(t)$. Thus, $\omega_1^\alpha(t) = \Theta_\alpha^{\mathbf{i}}(y)$, as required. □

Introduce a probability measure $\mathbb{P}_\mu$ under which π is a two-sided Brownian motion in V with drift $\mu \in C$. Set $\psi = (\pi(t), t \geqslant 0)$.

Proposition 5.9. *Under $\mathbb{P}_\mu$, the following statements hold:*

(1) *$\mathcal{Q}_{w_0}\pi$ has the same law as π.*
(2) *For each $t \in \mathbb{R}$, the random variables $\omega_1(t), \dots, \omega_q(t)$ are mutually independent and exponentially distributed with parameters $2\beta_1^\vee(\mu), \dots, 2\beta_q^\vee(\mu)$.*
(3) *For each $t \in \mathbb{R}$, $\omega(t)$ is independent of $(\mathcal{Q}_{w_0}\pi(s),\ -\infty < s \leqslant t)$.*
(4) *The random variables $\inf_{u\geqslant 0}\alpha^\vee(\mathcal{Q}_{w_0}\pi(u))$, α a simple root, are independent of the σ-algebra generated by $(\pi(t),\ t \geqslant 0)$.*

Proof. We see by backward induction on $k = q, \dots, 1$ that $\mathcal{Q}_{\beta_k} \dots \mathcal{Q}_{\beta_q}\pi(s)$, $s \leqslant t$, has the same distribution as $\mathcal{Q}_{\beta_{k-1}} \dots \mathcal{Q}_{\beta_q}\pi(s)$, $s \leqslant t$, is independent of $\omega_k(t)$, and that $\omega_k(t)$ has an exponential distribution with parameter $2\beta_k^\vee(\mu)$. At each step, this is a one-dimensional statement which can be checked directly or seen as a consequence of the classical output theorem for the $M/M/1$ queue (see, for example, [28]). This implies that (1), (2), and (3) hold. Moreover

$$\inf_{t \geqslant 0} \beta_1^\vee\big(\mathcal{Q}_{w_0}\pi(t)\big) = -\inf_{s \leqslant 0} \beta_1^\vee\big(\mathcal{Q}_{\beta_2} \dots \mathcal{Q}_{\beta_q}\pi(s)\big)$$

is independent of $\pi(t)$, $t \geqslant 0$. Since β_1 can be chosen as any simple root α, this proves (4). □

Let $T > 0$. For $\xi \in C$, denote by E_ξ the event that $\mathcal{Q}_{w_0}\pi(s) \in C - \xi$ for all $s \geqslant 0$ and by $E_{\xi,T}$ the event that $\mathcal{Q}_{w_0}\pi(s) \in C - \xi$ for all $T \geqslant s \geqslant 0$. By Proposition 5.9, E_ξ is independent of ψ.

For $r > 0$, define

$$B(\lambda, r) = \big\{\zeta \in V\colon \|\zeta - \lambda\| < r\big\}$$

and

$$R(z, r) = (z_1 - r, z_1 + r) \times \dots \times (z_q - r, z_q + r).$$

Fix (z, λ) in the interior of $L_{\mathbf{i}}$ and choose $\epsilon > 0$ sufficiently small so that $R(z, \epsilon)$ is contained in $L_{\mathbf{i}} \times B(\lambda, \epsilon)$ and

$$\inf_{\lambda' \in B(\lambda,\epsilon),\, z' \in R(z,\epsilon)} \alpha^\vee(\lambda') - \Theta_\alpha^{\mathbf{i}}(z') \geqslant 0. \tag{5.14}$$

Lemma 5.10.

$$\begin{aligned}
&\mathbb{P}_\mu\big(\mathcal{Q}_{w_0}\psi(T) \in B(\lambda, \epsilon),\ \varsigma_{\mathbf{i}}(\psi) \in R(z, \epsilon)\big)\\
&\quad = \lim_{C \ni \xi \to 0} \mathbb{P}_\mu(E_\xi)^{-1}\mathbb{P}_\mu\big(\mathcal{Q}_{w_0}\pi(T) \in B(\lambda, \epsilon),\ \omega(T) \in R(z, \epsilon), E_{\xi,T}\big).
\end{aligned}$$

Proof. An elementary induction argument on the recursive construction of $\mathcal{Q}_{w_0}$ shows that, on the event E_ξ, there is a constant C for which

$$\max_{i \leqslant q}\big\|y_i - \omega_i(T)\big\| \vee \big\|\mathcal{Q}_{w_0}\psi(T) - \mathcal{Q}_{w_0}\pi(T)\big\| \leqslant C\|\xi\|.$$

Hence, for ξ sufficiently small,

$$\begin{aligned}
&\mathbb{P}_\mu\big(\mathcal{Q}_{w_0}\psi(T) \in B\big(\lambda, \epsilon - C\|\xi\|\big),\ \varsigma_{\mathbf{i}}(\psi) \in R\big(z, \epsilon - C\|\xi\|\big),\ E_\xi\big)\\
&\quad \leqslant \mathbb{P}_\mu\big(\mathcal{Q}_{w_0}\pi(T) \in B(\lambda, \epsilon),\ \omega(T) \in R(z, \epsilon),\ E_\xi\big)\\
&\quad \leqslant \mathbb{P}_\mu\big(\mathcal{Q}_{w_0}\psi(T) \in B\big(\lambda, \epsilon + C\|\xi\|\big),\ \varsigma_{\mathbf{i}}(\psi) \in R\big(z, \epsilon + C\|\xi\|\big),\ E_\xi\big).
\end{aligned}$$

Now E_ξ is independent of ψ, and so

$$\begin{aligned}&\mathbb{P}_\mu\big(\mathcal{Q}_{w_0}\psi(T)\in B\big(\lambda,\epsilon-C\|\xi\|\big),\ \varsigma_{\mathbf{i}}(\psi)\in R\big(z,\epsilon-C\|\xi\|\big)\big)\\&\quad\leqslant \mathbb{P}_\mu(E_\xi)^{-1}\mathbb{P}_\mu\big(\mathcal{Q}_{w_0}\pi(T)\in B(\lambda,\epsilon),\ \omega(T)\in R(z,\epsilon),\ E_\xi\big)\\&\quad\leqslant \mathbb{P}_\mu\big(\mathcal{Q}_{w_0}\psi(T)\in B\big(\lambda,\epsilon+C\|\xi\|\big),\ \varsigma_{\mathbf{i}}(\psi)\in R\big(z,\epsilon+C\|\xi\|\big)\big).\end{aligned}$$

Letting $\xi\to 0$, we obtain that

$$\begin{aligned}&\mathbb{P}_\mu\big(\mathcal{Q}_{w_0}\psi(T)\in B(\lambda,\epsilon),\ \varsigma_{\mathbf{i}}(\psi)\in R(z,\epsilon)\big)\\&\quad=\lim_{C\ni\xi\to 0}\mathbb{P}_\mu(E_\xi)^{-1}\mathbb{P}_\mu\big(\mathcal{Q}_{w_0}\pi(T)\in B(\lambda,\epsilon),\ \omega(T)\in R(z,\epsilon),E_\xi\big).\end{aligned}\tag{5.15}$$

Finally observe that, on the event

$$\big\{\mathcal{Q}_{w_0}\pi(T)\in B(\lambda,\epsilon),\ \omega(T)\in R(z,\epsilon)\big\},$$

we have, by Lemma 5.8 and (5.14),

$$\begin{aligned}\inf_{u\geqslant T}\alpha^\vee\big(\mathcal{Q}_{w_0}\pi(u)\big)&=\alpha^\vee\big(\mathcal{Q}_{w_0}\pi(T)\big)-\Theta^{\mathbf{i}}_\alpha\big(\omega(T)\big)\\&\geqslant\inf_{\lambda'\in B(\lambda,\epsilon),\,z'\in R(z,\epsilon)}\alpha^\vee(\lambda')-\Theta^{\mathbf{i}}_\alpha(z')\geqslant 0.\end{aligned}$$

Thus, we can replace E_ξ by $E_{\xi,T}$ on the right-hand side of (5.15), and this concludes the proof of the lemma. □

For $a,b\in C$, define $\phi(a,b)=\sum_{w\in W}\varepsilon(w)e^{\langle wa,b\rangle}$.

Lemma 5.11. *Fix $\mu\in C$. The functions $f(a,b)=\phi(a,b)/[h(a)h(b)]$ and $g_\mu(a,b)=\phi(a,b)/\phi(a,\mu)$ have unique analytic extensions to $V\times V$. Moreover, $f(0,b)=k^{-1}$ and $g_\mu(0,b)=h(b)/h(\mu)$.*

Proof. It is clear that the function ϕ is analytic in (a,b), furthermore it vanishes on the hyperplanes $\langle\beta,a\rangle=0$, $\langle\beta,b\rangle=0$, for all roots β. The first claim follows from an elementary analytic functions argument. In the expansion of ϕ as an entire function, the term of homogeneous degree d is a polynomial in a,b which is antisymmetric under W, therefore a multiple of $h(a)h(b)$. In particular the term of lowest degree is a constant multiple of $h(a)h(b)$. This constant is nonzero, as can be seen by taking derivatives in the definition of ϕ. By l'Hôpital's rule, $\lim_{a\to 0}g_\mu(a,b)=h(b)/h(\mu)$. It follows that $\lim_{a\to 0}f(a,b)$ is a constant. To evaluate this constant, note that, since h is harmonic and vanishes at the boundary of C,

$$\int_C h(\lambda)^2e^{-\|\lambda\|^2/2}f(a,\lambda)\,d\lambda=e^{|a|^2/2}\int_V e^{-\|\lambda\|^2/2}\,d\lambda.$$

Letting $a\to 0$, we deduce that $f(0,\lambda)=k^{-1}$, as required. □

Denote by F_ξ the event that $\psi(s)\in C-\xi$ for all $s\geqslant 0$ and by $F_{\xi,T}$ the event that $\psi(s)\in C-\xi$ for all $T\geqslant s\geqslant 0$.

Lemma 5.12. *For $B \subset C$, bounded and measurable,*

$$\lim_{C\ni\xi\to 0} \mathbb{P}_\mu(F_\xi)^{-1}\mathbb{P}_\mu\bigl(\psi(T)\in B,\ F_{\xi,T}\bigr) = c_T^{-1} h(\mu)^{-1} \int_B e^{\langle\mu,\lambda\rangle - \|\mu\|^2 T/2} e^{-\|\lambda\|^2/2T} h(\lambda)\, d\lambda.$$

Proof. Set $z_T = \int_V e^{-\|\lambda\|^2/2T}\, d\lambda$. By the reflection principle,

$$\mathbb{P}_\mu\bigl(\psi(T)\in d\lambda,\ F_{\xi,T}\bigr) = e^{\langle\mu,\lambda\rangle - \|\mu\|^2 T/2} \sum_{w\in W} \varepsilon(w) p_T(w\xi, \xi+\lambda)\, d\lambda,$$

where $p_t(a,b) = z_t^{-1} e^{-\|b-a\|^2/2t}$ is the transition density of a standard Brownian motion in V. Integrating over λ and letting $T\to\infty$, we obtain (see [3])

$$\mathbb{P}_\mu(F_\xi) = \sum_{w\in W} \varepsilon(w) e^{\langle w\xi - \xi, \mu\rangle}.$$

Thus, using Lemma 5.11 and the bounded convergence theorem,

$$\begin{aligned}
&\lim_{C\ni\xi\to 0} \mathbb{P}_\mu(F_\xi)^{-1}\mathbb{P}_\mu\bigl(\psi(T)\in B,\ F_{\xi,T}\bigr)\\
&\quad = z_T^{-1} \lim_{C\ni\xi\to 0} \int_B e^{\langle\mu,\lambda\rangle - \|\mu\|^2 T/2} e^{-(|\xi|^2 + |\xi+\lambda|^2)/2T} \phi(\xi,\mu)^{-1} \phi\left(\xi, \frac{\xi+\lambda}{T}\right) d\lambda\\
&\quad = z_T^{-1} \lim_{C\ni\xi\to 0} \int_B e^{\langle\mu,\lambda\rangle - \|\mu\|^2 T/2} e^{-(\|\xi\|^2 + \|\xi+\lambda\|^2)/2T} g_\mu\left(\xi, \frac{\xi+\lambda}{T}\right) d\lambda\\
&\quad = z_T^{-1} h(\mu)^{-1} \int_B e^{\langle\mu,\lambda\rangle - \|\mu\|^2 T/2} e^{-|\lambda|^2/2T} h(\lambda/T)\, d\lambda\\
&\quad = c_T^{-1} h(\mu)^{-1} \int_B e^{\langle\mu,\lambda\rangle - \|\mu\|^2 T/2} e^{-\|\lambda\|^2/2T} h(\lambda)\, d\lambda,
\end{aligned}$$

as required. □

Applying Lemmas 5.10, 5.12 and Proposition 5.9, we obtain

$$\begin{aligned}
&\mathbb{P}_\mu\bigl(\mathcal{Q}_{w_0}\psi(T)\in B(\lambda,\epsilon),\ \varsigma_{\mathbf{i}}(\psi)\in R(z,\epsilon)\bigr)\\
&\quad = \lim_{C\ni\xi\to 0} \mathbb{P}_\mu(E_\xi)^{-1}\mathbb{P}_\mu\bigl(\mathcal{Q}_{w_0}\pi(T)\in B(\lambda,\epsilon),\ \omega(T)\in R(z,\epsilon), E_{\xi,T}\bigr) \quad \text{(Lemma 5.14)}\\
&\quad = \lim_{C\ni\xi\to 0} \mathbb{P}_\mu(E_\xi)^{-1}\mathbb{P}_\mu\bigl(\omega(T)\in R(z,\epsilon)\bigr)\mathbb{P}_\mu\bigl(\mathcal{Q}_{w_0}\pi(T)\in B(\lambda,\epsilon),\ E_{\xi,T}\bigr) \quad \text{(Lemma 5.9(3))}\\
&\quad = \lim_{C\ni\xi\to 0} \mathbb{P}_\mu(E_\xi)^{-1}\mathbb{P}_\mu\bigl(\omega(T)\in R(z,\epsilon)\bigr)\mathbb{P}_\mu\bigl(\psi(T)\in B(\lambda,\epsilon),\ F_{\xi,T}\bigr)\\
&\quad = \prod_{i=1}^q e^{-\beta_i^\vee(\mu) z_i}\bigl[e^{\epsilon\beta_i^\vee(\mu)} - e^{-\epsilon\beta_i^\vee(\mu)}\bigr]
\end{aligned}$$

$$\times \lim_{C\ni\xi\to 0} \mathbb{P}_\mu(E_\xi)^{-1}\mathbb{P}_\mu\big(\psi(T)\in B(\lambda,\epsilon),\ F_{\xi,T}\big) \quad \text{(Lemma 5.9(2))}$$

$$= \prod_{i=1}^{q} e^{-\beta_i^\vee(\mu)z_i}\big[e^{\epsilon\beta_i^\vee(\mu)} - e^{-\epsilon\beta_i^\vee(\mu)}\big]$$

$$\times c_T^{-1}h(\mu)^{-1} \int\limits_{B_V(\lambda,\epsilon)} e^{\mu(\lambda')-\|\mu\|^2T/2}e^{-\|\lambda'\|^2/2T}h(\lambda')\,d\lambda' \quad \text{(Lemma 5.12).}$$

Now divide by $\|B(y,\epsilon)\|(2\epsilon)^q$ and let ϵ tend to zero to obtain

$$\mathbb{P}_\mu\big(\mathcal{Q}_{w_0}\psi(T)\in d\lambda,\ \varsigma_{\mathbf{i}}(\psi)\in dz\big) = \prod_{i=1}^{q} e^{-\beta_i^\vee(\mu)z_i}e^{\langle\mu,\lambda\rangle-\|\mu\|^2T/2}c_T^{-1}h(\lambda)e^{-\|\lambda\|^2/2T}\,d\lambda\,dz.$$

Letting $\mu\to 0$ this becomes, writing $\mathbb{P}=\mathbb{P}_0$,

$$\mathbb{P}\big(\mathcal{Q}_{w_0}\psi(T)\in d\lambda,\ \varsigma_{\mathbf{i}}(\psi)\in dz\big) = c_T^{-1}h(\lambda)e^{-\|\lambda\|^2/2T}\,d\lambda\,dz. \tag{5.16}$$

Using Lemma 5.7, it follows that, for (w,λ) in the interior of $M_{\mathbf{i}}$,

$$\mathbb{P}\big(\varrho_{\mathbf{i}}(\eta)\in dw,\ \mathcal{P}_{w_0}\eta(T)\in d\lambda\big) = c_T^{-1}h(\lambda)e^{-\|\lambda\|^2/2T}\,dw\,d\lambda. \tag{5.17}$$

Under the probability measure $\mathbb{P}$, η is a standard Brownian motion in V with transition density given by $p_t(a,b)=z_t^{-1}e^{-\|b-a\|^2/2t}$. By Theorem 5.1 under $\mathbb{P}$, $\mathcal{P}_{w_0}\eta$ is a Brownian motion in C. Its transition density is given, for $\xi,\lambda\in C$, by

$$q_t(\xi,\lambda) = \frac{h(\lambda)}{h(\xi)}\sum_{w\in W}\varepsilon(w)p_t(w\xi,\lambda).$$

As remarked in [3], this transition density can be extended by continuity to the boundary of C. From Lemma 5.11 we see that $q_T(0,\lambda)=k^{-1}h(\lambda)^2e^{-\|\lambda\|^2/2T}$. Thus,

$$\mathbb{P}\big(\mathcal{P}_{w_0}\eta(T)\in d\lambda\big) = k^{-1}h(\lambda)^2e^{-\|\lambda\|^2/2T}\,d\lambda. \tag{5.18}$$

To complete the proof of the theorem, first note that since $\varsigma_{\mathbf{i}}(\psi)$ is measurable with respect to the σ-algebra generated by $(\mathcal{Q}_{w_0}\psi(u),\ u\geqslant T)$, $\varrho_{\mathbf{i}}(\eta)$ is measurable with respect to the σ-algebra generated by $(\mathcal{P}_{w_0}\eta(u),\ u\geqslant T)$. Thus, by the Markov property of $\mathcal{P}_{w_0}\eta$, the conditional distribution of $\varrho_{\mathbf{i}}(\eta)$, given $(\mathcal{P}_{w_0}\eta(s), s\leqslant T)$, is measurable with respect to the σ-algebra generated by $\mathcal{P}_{w_0}\eta(T)$. Combining this with (5.17) and (5.18), we conclude that the conditional law of $\varrho_{\mathbf{i}}(\eta)$, given $(\mathcal{P}_{w_0}\eta(s),\ s\leqslant T)$ and $\mathcal{P}_{w_0}\eta(t)=\lambda$, is almost surely uniform on $M_{\mathbf{i}}^\lambda$, and that the Euclidean volume of $M_{\mathbf{i}}^\lambda$ is $k^{-1}h(\lambda)$, as required.

5.5. Proof of Theorem 5.5

Let $\psi = w_0\eta$ and $\mathcal{Q}_{w_0} = \mathcal{P}_{w_0} w_0$. Denote by P_t (respectively Q_t) the semigroup of Brownian motion in V (respectively C). Under $\mathbb{P}$, by [3, Theorem 5.6], $\mathcal{Q}_{w_0}\psi$ is a Brownian motion in C. Let $\delta \in C$. The function $e_\delta(v) = e^{\langle \delta, v\rangle}$ is an eigenfunction of P_t and the e_δ-transform of P_t is a Brownian motion with drift δ. Setting $\phi_\delta(v) = \sum_{w\in W} \varepsilon(w) e^{\langle w\delta, v\rangle}$, the function ϕ_δ/h is an eigenfunction of Q_t and the (ϕ_δ/h)-transform of Q_t is a Brownian motion with drift δ conditioned never to exit C (see [3, Section 5.2] for a definition of this process). By Theorem 5.2, the conditional law of $\eta(T)$, given $(\mathcal{P}_{w_0}\eta(s),\ s \leqslant T)$ and $\mathcal{P}_{w_0}\eta(T) = \lambda$, is almost surely given by $\mu^\lambda_{\mathrm{DH}}$. It follows that the conditional law of $\psi(T)$, given $(\mathcal{Q}_{w_0}\psi(s),\ s \leqslant T)$ and $\mathcal{Q}_{w_0}\psi(T) = \lambda$, is almost surely given by $\mu^\lambda_{\mathrm{DH}}$. Denote the corresponding Markov operator by $K(\lambda, \cdot) = \mu^\lambda_{\mathrm{DH}}(\cdot)$. By [3, Theorem 5.6] we automatically have the intertwining $KP_t = Q_t K$. Note that Ke_δ is an eigenfunction of Q_t. By construction, the Ke_δ-transform of Q_t, started from the origin, has the same law as $\mathcal{Q}_{w_0}\psi^{(\delta)}$, where $\psi^{(\delta)}$ is a Brownian motion in V with drift δ. Recalling the proof of [3, Theorem 5.6] we note that $\mathcal{Q}_{w_0}\psi^{(\delta)}$ has the same law as a Brownian motion with drift δ conditioned never to exit C. It follows that $Ke_\delta = \phi_\delta/(c(\delta)h)$, for some $c(\delta) \neq 0$. Now observe (using Lemma 5.11 for example) that $\lim_{\xi\to 0} Ke_\delta(\xi) = 1$. Thus, by Lemma 5.11, $c(\delta) = \lim_{\xi\to 0} \phi_\delta(\xi)/h(\xi) = k^{-1}h(\delta)$. We conclude that

$$\int_V e^{\langle \delta, v\rangle} \mu^\lambda_{\mathrm{DH}}(dv) = k\frac{\sum_{w\in W} \varepsilon(w) e^{\langle w\delta, \lambda\rangle}}{h(\delta)h(\lambda)}.$$

This formula extends to $\delta \in V^*$ by analytic continuation (see Lemma 5.11 again), and the proof is complete.

5.6. A Littlewood–Richardson property

In usual Littelmann path theory, the concatenation of paths is used to describe tensor products of representations, and give a combinatorial formula for the Littlewood–Richardson coefficients. In our setting of continuous crystals, the representation theory does not exist in general, and the analogue of the Littlewood–Richardson coefficients is a certain conditional distribution of the Brownian path. In this section we describe this distribution in Theorem 5.15.

Let $\mathbf{i} = (s_1, \ldots, s_q)$ where $w_0 = s_1 \ldots s_q$ is a reduced decomposition. For $\eta \in C^0_T(V)$, let $x = \rho_{\mathbf{i}}(\eta)$.

For each simple root α choose now $\mathbf{j}_\alpha = (s^\alpha_1, \ldots, s^\alpha_q)$, a reduced decomposition of w_0, such that $s^\alpha_q = s_\alpha$, and denote the corresponding string parameters of the path η by $(\tilde{x}^\alpha_1, \ldots, \tilde{x}^\alpha_q) = \varrho_{\mathbf{j}_\alpha}(\eta)$. As in (5.2), there is a continuous function $\Psi'_\alpha : \mathbb{R}^q \to \mathbb{R}$ such that $\tilde{x}^\alpha_q = \Psi'_\alpha(x)$. Fix $\lambda, \mu \in C$ and suppose that $\lambda + \eta(s) \in C$ for $0 \leqslant s \leqslant T$. Then $\tilde{x}^\alpha_q = -\inf_{s\leqslant T} \alpha^\vee(\eta(s)) \leqslant \alpha^\vee(\lambda)$. In other words,

$$\Psi'_\alpha(x) \leqslant \alpha^\vee(\lambda), \quad \alpha \in \Sigma. \tag{5.19}$$

Let $M_{\mathbf{i}}^{\lambda,\mu}$ denote the set of $x \in M_{\mathbf{i}}^{\mu}$ which satisfy the additional constraints (5.19). This is a compact convex polytope. Let $\nu^{\lambda,\mu}$ be the uniform probability distribution on $M_{\mathbf{i}}^{\lambda,\mu}$ and let $\nu_{\lambda,\mu}$ be its image on V by the map

$$x = (x_1, \dots, x_q) \in M_{\mathbf{i}}^{\lambda,\mu} \mapsto \lambda + \mu - \sum_{j=1}^{q} x_j \alpha_j \in V.$$

Let η be the Brownian motion in V starting from 0. Observe that, by Theorem 3.12, the event $\{\eta(s) \in C - \lambda,\ 0 \leqslant s \leqslant T\}$ is measurable with respect to the σ-algebra generated by $\rho_{\mathbf{i}}(\eta)$. Combining this with Theorem 5.2 we obtain:

Corollary 5.13. *The conditional law of $\rho_{\mathbf{i}}(\eta)$, given $\mathcal{P}_{w_0}\eta(s)$, $s \leqslant T$, $\mathcal{P}_{w_0}\eta(T) = \mu$ and $\lambda + \eta(s) \in C$ for $0 \leqslant s \leqslant T$, is $\nu^{\lambda,\mu}$ and the conditional law of $\lambda + \eta(T)$ is $\nu_{\lambda,\mu}$.*

For $s, t \geqslant 0$ let

$$(\tau_s \eta)(t) = \eta(s+t) - \eta(s),$$
$$(\tau_s \mathcal{P}_{w_0}\eta)(t) = \mathcal{P}_{w_0}\eta(s+t) - \mathcal{P}_{w_0}\eta(s).$$

Lemma 5.14. *For all $s \geqslant 0$,*

$$\mathcal{P}_{w_0}(\tau_s \mathcal{P}_{w_0}\eta) = \mathcal{P}_{w_0}\tau_s\eta.$$

Proof. If $\pi_1, \pi_2 : \mathbb{R}^+ \to V$ are continuous path starting at 0, let $\pi_1 \star_s \pi_2$ be the path defined by $\pi_1 \star_s \pi_2(r) = \pi_1(r)$ when $0 \leqslant r \leqslant s$ and $\pi_1 \star_s \pi_2(r) = \pi_1(s) + \pi_2(r-s)$ when $s \leqslant r$. By Lemma 4.12, $\mathcal{P}_{w_0}(\pi_1 \star_s \pi_2) = \mathcal{P}_{w_0}(\pi_1) \star_s \tilde{\pi}_2$ where $\tilde{\pi}_2$ is a path such that $\mathcal{P}_{w_0}(\tilde{\pi}_2) = \mathcal{P}_{w_0}(\pi_2)$. Since $\tau_s(\pi_1 \star_s \pi_2) = \pi_2$, this gives the lemma. □

Let $\gamma_{\lambda,\mu}$ be the measure on C given by

$$\gamma_{\lambda,\mu}(dx) = \frac{h(x)}{h(\lambda)}\, \nu_{\lambda,\mu}(dx).$$

It will follow from Theorem 5.15 that this is a probability measure. Consider the following σ-algebra

$$\mathcal{G}_{s,t} = \sigma\big(\mathcal{P}_{w_0}\eta(a),\ a \leqslant s,\ \mathcal{P}_{w_0}\tau_s\eta(r),\ r \leqslant t\big).$$

The following result is a continuous analogue of the Littelmann interpretation of the Littlewood–Richardson decomposition of a tensor product.

Theorem 5.15. *For $s, t > 0$, $\gamma_{\lambda,\mu}$ is the conditional distribution of $\mathcal{P}_{w_0}\eta(s+t)$ given $\mathcal{G}_{s,t}$, $\mathcal{P}_{w_0}\eta(s) = \lambda$ and $\mathcal{P}_{w_0}\tau_s\eta(t) = \mu$.*

Proof. When $(X_t, (\theta_t), \mathbb{P}_x)$ is a Markov process with shift θ_t (i.e. $X_{s+t} = X_s \circ \theta_t$), for any $\sigma(X_r, r \geqslant 0)$-measurable random variables $Z, Y \geqslant 0$, one has

$$\mathbb{E}\big(Z \circ \theta_t \big| \sigma(X_s,\ s \leqslant t, Y \circ \theta_t)\big) = \mathbb{E}_{X_0}\big(Z \big| \sigma(Y)\big) \circ \theta_t.$$

Let us apply this relation to the Markov process $X = \mathcal{P}_{w_0}\eta$ (see [3]). Notice that since $\mathcal{P}_{w_0}(\tau_s X) = \mathcal{P}_{w_0}(\tau_0 X) \circ \theta_s$, it follows from the lemma that

$$\mathcal{G}_{s,t} = \sigma\big(X_a, \mathcal{P}_{w_0}(\tau_0 X)(r) \circ \theta_s,\ a \leqslant s,\ r \leqslant t\big).$$

Therefore, for any Borel nonnegative function $f : V \to \mathbb{R}$,

$$\mathbb{E}\big[f\big(\mathcal{P}_{w_0}\eta(s+t)\big)\big|\mathcal{G}_{s,t}\big] = \mathbb{E}_{X_0}\big[f(X_t)|\sigma\big(\mathcal{P}_{w_0}(\tau_0 X)(r),\ r \leqslant t\big)\big] \circ \theta_s.$$

One knows [3, Theorem 5.1] that X is the h-process of the Brownian motion killed at the boundary of C. In other words, starting from $X_0 = \lambda$, X is the h-process of $\lambda + \eta(t)$ conditionally on $\lambda + \eta(s) \in C$, for $0 \leqslant s \leqslant t$. It thus follows from Corollary 5.13 that

$$\mathbb{E}_\lambda\big[f(X_t)\big|\sigma\big(\mathcal{P}_{w_0}(\tau_0 X)(r),\ r \leqslant t\big)\big] = \frac{1}{h(\lambda)} \int f(x) h(x)\, d\nu_{\lambda,\mu}(x)$$

when $\mathcal{P}_{w_0}(\tau_0 X)(t) = \mu$. This proves that

$$\mathbb{E}\big[f\big(\mathcal{P}_{w_0}\eta(s+t)\big)\big|\mathcal{G}_{s,t}\big] = \int f(x)\, d\mu_{\lambda,\mu}(x)$$

when $\mathcal{P}_{w_0}\eta(s) = \lambda$ and $\mathcal{P}_{w_0}\tau_s\eta(t) = \mu$. $\square$

5.7. A product formula

Consider the Laplace transform of $\mu_{\mathrm{DH}}^\lambda$ given, for $\lambda \in C,\ z \in V^*$, by

$$J_\lambda(z) = k \frac{\sum_W \varepsilon(w) e^{\langle z, w\lambda\rangle}}{h(z)h(\lambda)}. \tag{5.20}$$

This is an example of a generalized Bessel function, following the terminology of Helgason [12] in the Weyl group case and Opdam [29] in the general Coxeter case. It was a conjecture in Gross and Richards [10] that these are Laplace transform of positive measures (this also follows from Rösler [31]). They are positive eigenfunctions of the Laplace and of the Dunkl operators on the Weyl chamber C with eigenvalue $\|\lambda\|^2$ and Dirichlet boundary conditions and $J_\lambda(0) = 1$. Let f_λ be the density of the probability measure $\mu_{\mathrm{DH}}^\lambda$. One has

$$\int_V e^{\langle z, v\rangle} f_\lambda(v)\, dv = J_\lambda(z). \tag{5.21}$$

Let, for $v \in C$,

$$f_{\lambda,\mu}(v) = \frac{1}{h(\mu)} \sum_{w \in W} h(wv) f_\lambda(wv - \mu).$$

It follows from the next result that $f_{\lambda,\mu}(v) \geqslant 0$.

Theorem 5.16.

(i) *For* $\lambda, \mu \in C$ *and* $z \in V^*$,

$$J_\lambda(z) J_\mu(z) = \int_C J_v(z) f_{\lambda,\mu}(v)\, dv.$$

(ii)
$$\gamma_{\lambda,\mu}(dx) = f_{\lambda,\mu}(x)\, dx.$$

Proof. The first part is given by the following computation, similar to the one in Dooley et al. [6], we give it for the convenience of the reader. It follows from (5.20) and (5.21) that

$$J_\lambda(z) J_\mu(z) = \int_V e^{\langle z,v\rangle} J_\mu(z) f_\lambda(v)\, dv = k \sum_W \varepsilon(w) \int_V \frac{e^{\langle z, w\mu+v\rangle}}{h(\mu)h(z)} f_\lambda(v)\, dv.$$

Using the invariance of the measure $\mu^\lambda_{\mathrm{DH}}$ under W, $f_\lambda(wv) = f_\lambda(v)$ for $w \in W$. One has

$$\begin{aligned}
J_\lambda(z) J_\mu(z) &= k \sum_W \varepsilon(w) \int_V \frac{e^{\langle z, w(\mu+v)\rangle}}{h(\mu)h(z)} f_\lambda(v)\, dv \\
&= k \sum_W \varepsilon(w) \int_V \frac{e^{\langle z, wv\rangle}}{h(\mu)h(z)} f_\lambda(v-\mu)\, dv \\
&= \frac{1}{h(\mu)} \int_V J_v(z) h(x) f_\lambda(v-\mu)\, dv \\
&= \frac{1}{h(\mu)} \sum_{w\in W} \int_{w^{-1}C} J_v(z) h(v) f_\lambda(v-\mu)\, dv \\
&= \frac{1}{h(\mu)} \sum_{w\in W} \int_C J_v(z) h(wv) f_\lambda(wv-\mu)\, dv \\
&= \int_C J_z(v) f_{\lambda,\mu}(v)\, dv
\end{aligned}$$

where we have used that, up to a set of measure zero, $V = \bigcup_{w\in W} w^{-1}C$. This proves (i).

Let us now prove (ii), using Theorem 5.15. Since η is a standard Brownian motion in V, $\{\eta(r),\ r \leqslant s\}$ and $\tau_s\eta$ are independent, hence, for $z \in V^*$,

$$\begin{aligned}
\mathbb{E}\big(e^{\langle z,\eta(s+t)\rangle}\big) \big| \mathcal{G}_{s,t}\big) &= \mathbb{E}\big(e^{\langle z,\eta(s)\rangle} e^{\langle z,\tau_s\eta(t)\rangle} \big| \mathcal{G}_{s,t}\big) \\
&= \mathbb{E}\big(e^{\langle z,\eta(s)\rangle} \big| \sigma\big(\mathcal{P}_{w_0}\eta(a),\ a \leqslant s\big)\big) \mathbb{E}\big(e^{\langle z,\tau_s\eta(t)\rangle} \big| \sigma\big(\mathcal{P}_{w_0}\tau_s\eta(b),\ b \leqslant t\big)\big).
\end{aligned}$$

By Theorem 5.5,

$$J_\lambda(z) = \mathbb{E}\big(e^{\langle z, \eta(s)\rangle} \big| \sigma\big(\mathcal{P}_{w_0}\eta(a),\ a \leqslant s\big)\big)$$

when $\mathcal{P}_{w_0}\eta(s) = \lambda$ and, since $\tau_s\eta$ and η have the same law,

$$J_\mu(z) = \mathbb{E}\big(e^{\langle z, \tau_s\eta(t)\rangle} \big| \sigma\big(\mathcal{P}_{w_0}\tau_s\eta(b),\ b \leqslant t\big)\big)$$

when $\mathcal{P}_{w_0}\tau_s\eta(t) = \mu$. Therefore

$$\mathbb{E}\big(e^{\langle z, \eta(s+t)\rangle} \big| \mathcal{G}_{s,t}\big) = J_\lambda(z) J_\mu(z).$$

On the other hand, by Lemma 4.12, $\mathcal{G}_{s,t}$ is contained in $\sigma(\mathcal{P}_{w_0}\eta(r), r \leqslant s+t)$, thus

$$\begin{aligned}\mathbb{E}\big(e^{\langle z, \eta(s+t)\rangle} \big| \mathcal{G}_{s,t}\big) &= \mathbb{E}\big(\mathbb{E}\big(e^{\langle z, \eta(s+t)\rangle} \big| \sigma\big(\mathcal{P}_{w_0}\eta(r),\ r \leqslant s+t\big)\big) \big| \mathcal{G}_{s,t}\big) \\ &= \mathbb{E}\big(J_z\big(\mathcal{P}_{w_0}\eta(s+t)\big) \big| \mathcal{G}_{s,t}\big).\end{aligned}$$

It thus follows from Theorem 5.15 that

$$J_\lambda(z) J_\mu(z) = \int J_v(z)\, d\gamma_{\lambda,\mu}(v).$$

Therefore, for all $z \in V^*$,

$$\int J_v(z) f_{\lambda,\mu}(v)\, dv = \int J_v(z)\, d\gamma_{\lambda,\mu}(v).$$

By injectivity of the Fourier–Laplace transform this implies that

$$d\gamma_{\lambda,\mu}(v) = f_{\lambda,\mu}(v)\, dv. \qquad \square$$

The positive product formula gives a positive answer to a question of Rösler [32] for the radial Dunkl kernel. It shows that one can generalize the structure of Bessel–Kingman hypergroup to any Weyl chamber, for the so called geometric parameter.

6. Littelmann modules and geometric lifting

6.1. It was observed some time ago by Lusztig that the combinatorics of the canonical basis is closely related to the geometry of the totally positive varieties. This connection was made precise by Berenstein and Zelevinsky in [2], in terms of transformations called "tropicalization" and "geometric lifting." In this section we show how some simple considerations on Sturm–Liouville equations lead to a natural way of lifting Littelmann paths, which take values in a Cartan algebra, to the corresponding Borel group. Using this lift, an application of Laplace's method explains the connection between the canonical basis and the totally positive varieties.

This section is organized as follows. We first recall the notions of tropicalization and geometric lifting in the next subsection, as well as the connection between the totally positive varieties and the canonical basis. Then we make some observations on Sturm–Liouville equations and their relation to Pitman transformations and the Littelmann path model in type A_1. We extend these observations to higher rank in the next subsections then we show, in Theorem 6.5 how they explain the link between string parametrization of the canonical basis and the totally positive varieties.

6.2. Tropicalization and geometric lifting

A subtraction free rational expression is a rational function in several variables, with positive real coefficients and without minus sign, e.g.

$$t_1 + 2t_2/t_3, \qquad \big(1 - t^3\big)/(1 - t) \quad \text{or} \quad 1/(t_1 t_2 + 3t_3 t_4)$$

are such expressions, but not $t_1 - t_2$. Any such expression $F(t_1, \ldots, t_n)$ can be tropicalized, which means that

$$F_{\text{trop}}(x_1, \ldots, x_n) = \lim_{\varepsilon \to 0_+} \varepsilon \log\big(F\big(e^{x_1/\varepsilon}, \ldots, e^{x_n/\varepsilon}\big)\big)$$

exists as a piecewise linear function of the real variables $(x_1, \ldots, x_n)$, and is given by an expression in the maxplus algebra over the variables $x_1, \ldots, x_n$. More precisely, the tropicalization $F \to F_{\text{trop}}$ replaces each occurrence of $+$ by $\vee$ (the max sign $x \vee y = \max(x, y)$), each product by a $+$, and each fraction by a $-$, and each positive real number by 0. For example the three expressions above give

$$(t_1 + 2t_2/t_3)_{\text{trop}} = x_1 \vee (x_2 - x_3), \qquad \big(\big(1 - x^3\big)/(1 - x)\big)_{\text{trop}} = 0 \vee x \vee 2x,$$

and

$$\big(1/(t_1 t_2 + 3t_3 t_4)\big)_{\text{trop}} = -\big((x_1 + x_2) \vee (x_3 + x_4)\big).$$

Tropicalization is not a one to one transformation, and there exists in general many subtraction free rational expressions which have the same tropicalization. Given some expression G in the maxplus algebra, any subtraction free rational expression whose tropicalization is G is called a geometric lifting of G, cf. [2].

6.3. Double Bruhat cells and string coordinates

We recall some standard terminology, using the notations of [2]. We consider a simply connected complex semisimple Lie group G, associated with a root system R. Let H be a maximal torus, and B, B_- be corresponding opposite Borel subgroups with unipotent radicals N, N_-. Let α_i, $i \in I$, and $\alpha_i^\vee$, $i \in I$, be the simple positive roots and coroots, and s_i the corresponding reflections in the Weyl group W. Let e_i, f_i, h_i, $i \in I$, be Chevalley generators of the Lie algebra of G. One can choose representatives $\overline{w} \in G$ for $w \in W$ by putting $\overline{s_i} = \exp(-e_i)\exp(f_i)\exp(-e_i)$ and $\overline{vw} = \bar{v}\,\overline{w}$ if $l(v) + l(w) = l(vw)$ (see [8, (1.8), (1.9)]). The Lie algebra of H, denoted by $\mathfrak{h}$ has a Cartan decomposition $\mathfrak{h} = \mathfrak{a} + i\mathfrak{a}$ such that the roots α_i take real values on the real vector space $\mathfrak{a}$. Thus $\mathfrak{a}$ is generated by $\alpha_i^\vee$, $i \in I$, and its dual $\mathfrak{a}^*$ by α_i, $i \in I$.

A double Bruhat cell is associated with each pair $u, v \in W$ as

$$L^{u,v} = N\bar{u}N \cap B_-\bar{v}B_-.$$

We will be mainly interested here in the double Bruhat cells $L^{w,e}$. As shown in [2], given a reduced decomposition $w = s_{i_1} \dots s_{i_q}$ every element $g \in L^{w,e}$ has a unique decomposition $g = x_{-i_1}(r_1) \dots x_{-i_q}(r_q)$ with nonzero complex numbers $(r_1, \dots, r_q)$, where $x_{-i}(s) = \varphi_i \left(\begin{smallmatrix} s & 0 \\ 1 & s^{-1} \end{smallmatrix}\right)$ (where φ_i is the embedding of SL_2 into G given by e_i, f_i, h_i). The totally positive part of the double Bruhat cell corresponds to the set of elements with positive real coordinates. For two different reduced decompositions, the transition map between two sets of coordinates of the form $(r_1, \dots, r_q)$ is given by a subtraction free rational map, which is therefore subject to tropicalization.

As a simple example consider the case of type A_2. Let the coordinates on the double Bruhat cell $L^{w_0,e}$ for the reduced decompositions $w_0 = s_1 s_2 s_1$, and $w_0 = s_2 s_1 s_2$ be (u_1, u_2, u_3) and (t_1, t_2, t_3) respectively, then

$$\begin{pmatrix} t_2 & 0 & 0 \\ t_1 & t_1 t_3/t_2 & 0 \\ 1 & t_3/t_2 + 1/t_1 & 1/t_1 t_3 \end{pmatrix} = \begin{pmatrix} u_1 u_3 & 0 & 0 \\ u_3 + u_2/u_1 & u_2/u_1 u_3 & 0 \\ 1 & 1/u_3 & 1/u_2 \end{pmatrix} \tag{6.1}$$

which yields transition maps

$$\begin{aligned} t_1 &= u_3 + u_2/u_1, \\ t_2 &= u_1 u_3, \\ t_3 &= u_1 u_2/(u_2 + u_1 u_3). \end{aligned} \tag{6.2}$$

On the other hand, for each reduced expression $w_0 = s_{i_1} \dots s_{i_q}$ we can consider the parametrization of the canonical basis by means of string coordinates. For any two such reduced decompositions, the transition maps between the two sets of string coordinates are given by piecewise linear expressions. As shown by Berenstein and Zelevinsky, these expressions are the tropicalizations of the transition maps between the two parametrizations of the double Bruhat cell $L^{w_0,e}$, associated to the Langlands dual group. For example, in type A_2 (which is its own Langlands dual) let (x_1, x_2, x_3) be the Kashiwara, or string, coordinates of the canonical basis, using the reduced decomposition $w_0 = s_1 s_2 s_1$, and (y_1, y_2, y_3) the ones corresponding to $w_0 = s_2 s_1 s_2$. The transition map between the two is given by

$$\begin{aligned} y_1 &= x_3 \vee (x_2 - x_1), \\ y_2 &= x_1 + x_3, \\ y_3 &= x_1 \wedge (x_2 - x_3) \end{aligned}$$

which is the tropicalization of (6.2).

We will show how some elementary considerations on the Sturm–Liouville equation, and the method of variation of constants, together with the Littelmann path model explain these connections.

6.4. Sturm–Liouville equations

We consider the Sturm–Liouville equation

$$\varphi'' + q\varphi = \lambda\varphi \tag{6.3}$$

on some interval of the real line, say $[0, T]$ to fix notations. In general there exists no closed form for the solution to such an equation. However, if one solution φ_0 is known, which does not vanish in the interval then all the solutions can be found by quadrature. Indeed using for example the "method of variation of constants" one sees that every other solution φ of this equation in the same interval can be written in the form

$$\varphi(t) = u\varphi_0(t) + v\varphi_0(t)\int_0^t \frac{1}{\varphi_0^2(s)}\,ds$$

for some constants u, v. If this new solution does not vanish in the interval I, we can use it to generate other solutions of the equation by the same kind of formula. This leads us to investigate the composition of two maps of the form

$$E_{u,v} : \varphi \mapsto u\varphi(t) + v\varphi(t)\int_0^t \frac{1}{\varphi^2(s)}\,ds$$

acting on nonvanishing continuous functions. It is easy to see, using integration by parts, that whenever the composition is well defined, one has

$$E_{u,v} \circ E_{u',v'} = E_{uu',uv'+v/u'};$$

therefore these maps define a partial right action of the group of unimodular lower triangular matrices

$$\begin{pmatrix} u & 0 \\ v & u^{-1} \end{pmatrix}$$

on the set of continuous paths which do not vanish in I. Of course this is equivalently a partial left action of the upper triangular group, but for reasons which will soon appear we choose this formulation. In particular if we start from φ and construct

$$\psi(t) = u\varphi(t) + v\varphi(t)\int_0^t \frac{1}{\varphi^2(s)}\,ds$$

which does not vanish on $[0, T]$, then φ can be recovered from ψ by the formula

$$\varphi(t) = u^{-1}\psi(t) - v\psi(t)\int_0^t \frac{1}{\psi^2(s)}\,ds.$$

Coming back to the Sturm–Liouville equation, let η, ρ be a fundamental basis of solutions at 0, namely $\eta(0) = \rho'(0) = 1$, $\eta'(0) = \rho(0) = 0$. Then in the two-dimensional space spanned by ρ, η the transformation is given by

$$(x, y) \mapsto (ux, uy + v/x)$$

and it is defined on $x \neq 0$. Again it is easy to check, using this formula, that this defines a right action of the lower triangular group.

Let us now investigate the limiting case $u = 0$, which gives (assuming $v = 1$ for simplicity)

$$\mathcal{T}\varphi(t) = \varphi(t) \int_0^t \frac{ds}{\varphi(s)^2}. \tag{6.4}$$

This map provides a "geometric lifting" of the one-dimensional Pitman transformation. Indeed set $\varphi(t) = e^{a(t)}$, then using Laplace's method

$$\lim_{\varepsilon \to 0_+} \varepsilon \log\left(e^{a(t)/\varepsilon} \int_0^t e^{-2a(s)/\varepsilon}\, ds\right) = a(t) - 2 \inf_{0 \leqslant s \leqslant t} a(s). \tag{6.5}$$

This time the function φ cannot be recovered from $\mathcal{T}\varphi$. If we compute the same transformation with $\varphi_v(t) := \varphi(t)(1 + v\int_0^t \frac{1}{\varphi(s)^2}\, ds)$ we get

$$\begin{aligned}
\mathcal{T}\varphi_v(t) &= \varphi_v(t) \int_0^t \frac{1}{\varphi_v(s)^2}\, ds \\
&= \varphi(t)\left(1 + v\int_0^t \frac{1}{\varphi(s)^2}\, ds\right)\left(\frac{1}{v} - \frac{1}{v(1 + v\int_0^t \frac{1}{\varphi(s)^2}\, ds)}\right) \\
&= \varphi(t) \int_0^t \frac{1}{\varphi(s)^2}\, ds \\
&= \mathcal{T}\varphi(t).
\end{aligned}$$

This is of course not surprising, since $\mathcal{T}\varphi$ vanishes at 0, it thus belongs to a one-dimensional subspace of the space of solutions to the Sturm–Liouville equation, and $\mathcal{T}$ is not invertible. In order to recover the function φ from $\psi = \mathcal{T}\varphi$ we thus need to specify some real number. A convenient choice is to impose the value of

$$\xi = \int_0^T \frac{1}{\varphi(s)^2}\, ds = \frac{\psi(T)}{\varphi(T)}.$$

With this we can compute

$$\int_t^T \frac{1}{\psi(s)^2}\, ds = \frac{1}{\int_0^t \frac{1}{\varphi(s)^2}\, ds} - \frac{1}{\int_0^T \frac{1}{\varphi(s)^2}\, ds} = \frac{\varphi(t)}{\psi(t)} - \frac{1}{\xi}.$$

Proposition 6.1. *Assume that $\psi = \mathcal{T}\varphi$ for some nonvanishing φ, then the set $\mathcal{T}^{-1}(\psi)$ can be parametrized by $\xi \in]0, +\infty[$. For each such ξ there exists a unique $\varphi_\xi \in \mathcal{T}^{-1}(\psi)$ such that $\xi = \int_0^T \frac{1}{\varphi_\xi(s)^2}\,ds$, given by*

$$\varphi_\xi(t) = \psi(t)\left(\frac{1}{\xi} + \int_t^T \frac{1}{\psi(s)^2}\,ds\right).$$

Identifying the positive half-line with the Weyl chamber for SL_2, we see that sets of the form $\mathcal{T}^{-1}(\psi)$ are geometric liftings of the Littelmann modules for SL_2. The formula in the proposition gives a geometric lifting of the operator $\mathcal{H}^x$ since

$$\mathcal{H}^x a(t) = a(t) - x \wedge 2 \inf_{t \leqslant s \leqslant T} a(s) = \lim_{\varepsilon \to 0_+} \varepsilon \log\left(e^{a(t)/\varepsilon}\left(e^{-x/\varepsilon} + \int_t^T e^{-2a(s)/\varepsilon}\,ds\right)\right).$$

We shall now find the geometric liftings of the Littelmann operators. For this we have, knowing an element $\varphi_{\xi_1} \in \mathcal{T}^{-1}(\psi)$, to find the solution corresponding to ξ_2. Since

$$\varphi_{\xi_i}(t) = \psi(t)\left(\frac{1}{\xi_i} + \int_t^T \frac{1}{\psi(s)^2}\,ds\right), \quad i = 1, 2,$$

one has

$$\varphi_{\xi_1} = \varphi_{\xi_2} + \psi\left(\frac{1}{\xi_1} - \frac{1}{\xi_2}\right) = \varphi_{\xi_2}\left(1 + \left(\frac{1}{\xi_1} - \frac{1}{\int_0^T \frac{1}{\varphi_{\xi_2}(s)^2}\,ds}\right)\int_0^t \frac{1}{\varphi_{\xi_2}(s)^2}\,ds\right).$$

Using Laplace method again one can recover the formula for the operators $\mathcal{E}_\alpha^x$, see Definition 3.3.

6.5. A 2×2 matrix interpretation

We shall now recast the above computations using a 2×2 matrix differential equation of order one, and the Gauss decomposition of matrices. This will allow us in the next section to extend these constructions to higher rank groups.

Let N_+ be the nilpotent group of upper triangular invertible 2×2 matrices, let N_- be the corresponding group of lower triangular matrices, and A the group of diagonal matrices, then an invertible 2×2 matrix g has a Gauss decomposition if it can be written as $g = [g]_-[g]_0[g]_+$ with $[g]_- \in N_-$, $[g]_0 \in A$ and $[g]_+ \in N_+$. We will use also the decomposition $g = [g]_-[g]_{0+}$ with $[g]_{0+} = [g]_0[g]_+ \in B = AN_+$. The condition for such a decomposition to exist is exactly that the upper left coefficient of the matrix g be nonzero.

Let us consider a smooth path $a : [0, T] \to \mathbb{R}$, such that $a(0) = 0$, and let the matrix $b(t)$ be the solution to

$$\frac{db}{dt} = \begin{pmatrix} \frac{da}{dt} & 1 \\ 0 & -\frac{da}{dt} \end{pmatrix} b; \qquad b(0) = Id. \tag{6.6}$$

Then one has

$$b(t)=\begin{pmatrix} e^{a(t)} & e^{a(t)}\int_0^t e^{-2a(s)}\,ds \\ 0 & e^{-a(t)} \end{pmatrix}.$$

Now let $g=\begin{pmatrix} u & 0 \\ v & u^{-1}\end{pmatrix}$ and consider the Gauss decomposition of the matrix

$$bg=\begin{pmatrix} ue^{a(t)}+ve^{a(t)}\int_0^t e^{-2a(s)}\,ds & u^{-1}e^{a(t)}\int_0^t e^{-2a(s)}\,ds \\ ve^{-a(t)} & u^{-1}e^{-a(t)} \end{pmatrix}.$$

One finds that

$$[bg]_-=\begin{pmatrix} 1 & 0 \\ \frac{ve^{-a(t)}}{ue^{a(t)}+ve^{a(t)}\int_0^t e^{-2a(s)}\,ds} & 1 \end{pmatrix}$$

and

$$[bg]_{0+}=\begin{pmatrix} ue^{a(t)}+ve^{a(t)}\int_0^t e^{-2a(s)}\,ds & u^{-1}e^{a(t)}\int_0^t e^{-2a(s)}\,ds \\ 0 & (ue^{a(t)}+ve^{a(t)}\int_0^t e^{-2a(s)}\,ds)^{-1} \end{pmatrix}.$$

One can check the following proposition.

Proposition 6.2. *The upper triangular matrix $[bg]_{0+}$ satisfies the differential equation*

$$\frac{d}{dt}[bg]_{0+}=\begin{pmatrix} \frac{d}{dt}T_{u,v}a(t) & 1 \\ 0 & -\frac{d}{dt}T_{u,v}a(t) \end{pmatrix}[bg]_{0+}$$

where $T_{u,v}a(t)=\log(E_{u,v}e^{a(t)})$.

This equation is of the same kind as Eq. (6.6) satisfied by the original matrix b, but with a different initial point. The right action $E_{u,v}$ is thus obtained by taking the matrix solution to (6.6), multiplying it on the right by $g=\begin{pmatrix} u & 0 \\ v & u^{-1}\end{pmatrix}$ and looking at the diagonal part of the Gauss decomposition of the resulting matrix. Actually in this way the partial action $T_{u,v}$ extends to a partial action T_g of the whole group of invertible real 2×2 matrices. One starts from the path a, constructs the matrix b by the differential equation and then takes the 0-part in the Gauss decomposition of bg. This yields a path $T_g a$. The statement of the proposition above remains true for $[bg]_{0+}$. The importance of this statement is that one can iterate the procedure and see that $T_{g_1g_2}=T_{g_2}\circ T_{g_1}$ when defined.

Consider now the element $s=\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$, then

$$T_s a(t)=a(t)+\log\left(\int_0^t e^{-2a(s)}\,ds\right).$$

This is the geometric lifting of the Pitman operator obtained in (6.4). In the next section we shall extend these considerations to groups of higher rank.

6.6. Paths in the Cartan algebra

We work now in the general framework of the beginning of Section 6.3.

One has the usual decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{a} + \mathfrak{n}_+$. Correspondingly there is a Gauss decomposition $g = [g]_-[g]_0[g]_+$ with $[g]_- \in N_-$, $[g]_0 \in A$, $[g]_+ \in N$, defined on an open dense subset. We denote by $[g]_{0+} = [g]_0[g]_+$ the $B = AN_+$ part of the decomposition.

The following is easy to check, and provides a useful characterization of the vector space generated by the e_i.

Lemma 6.3. *Let $n \in \mathfrak{n}_+$, then one has $[h^{-1}nh]_+ = n$ for all $h \in N_-$ if and only if n belongs to the vector space generated by the e_i.*

Let a be a path in the Cartan algebra $\mathfrak{a}$ and let b be a solution to the equation

$$\frac{d}{dt}b = \left(\frac{d}{dt}a + n\right)b$$

where $n \in \bigoplus_i \mathbb{C}e_i$.

Proposition 6.4. *Let $g \in G$, and assume that bg has a Gauss decomposition, then the upper part $[bg]_{0+}$ in the Gauss decomposition of bg satisfies the equation*

$$\frac{d}{dt}[bg]_{0+} = \left(\frac{d}{dt}T_g a + n\right)[bg]_{0+} \tag{6.7}$$

where $T_g a(t)$ is a path in the Cartan algebra.

Proof. Let us write the equation

$$\frac{d}{dt}\big([bg]_-[bg]_{0+}\big) = \left(\frac{d}{dt}a + n\right)[bg]_-[bg]_{0+}$$

in the form

$$[bg]_-^{-1}\frac{d}{dt}[bg]_- = [bg]_-^{-1}\left(\frac{d}{dt}a + n\right)[bg]_- - \frac{d}{dt}[bg]_{0+}[bg]_{0+}^{-1}.$$

Since the left-hand side of this equation is lower triangular, the right-hand side has zero upper triangular part therefore, by Lemma 6.3

$$n = \left[[bg]_-^{-1}\left(\frac{d}{dt}a + n\right)[bg]_-\right]_+ = \left[\frac{d}{dt}[bg]_{0+}[bg]_{0+}^{-1}\right]_+$$

therefore there exists a path $T_g a$ such that Eq. (6.7) holds. □

We now assume that

$$n = \sum_i n_i e_i$$

with all $n_i > 0$. When $g = \bar{s}_i$ is a fundamental reflection, one gets a geometric lifting of the Pitman operator

$$T_{s_i} a(t) = a(t) + \log\left(\int_0^t e^{-\alpha_i(a(s))} \, ds \right) \alpha_i^\vee$$

associated with the dual root system, i.e.

$$\lim_{\varepsilon \to 0} \varepsilon T_{s_i} \left(\frac{1}{\varepsilon} a \right) = \mathcal{P}_{\alpha_i^\vee} a.$$

Thanks to the above proposition, one can prove that these geometric liftings satisfy the braid relations, and T_w provides a geometric lifting of the Pitman operator $\mathcal{P}_w$ for all $w \in W$.

Analogously the Littelmann raising and lowering operators also have geometric liftings.

6.7. Reduced double Bruhat cells

In this section we show how our considerations on Littelmann's path model allow us to make the connection with the work of Berenstein and Zelevinsky [2]. We consider a path a on the Cartan Lie algebra, with $a(0) = 0$, then belongs to the Littelmann module $L_{\mathcal{P}_{w_0} a}$.

Consider the solution b to $\frac{d}{dt} b = (\frac{d}{dt} a + n) b$, $b(0) = I$. Then $[[b]_+ w_0]_{-0} \in L^{w_0, e}$, thus if

$$w_0 = s_{i_1} \dots s_{i_q} \tag{6.8}$$

is a reduced decomposition, then one has

$$\big[[b]_+ w_0\big]_{-0} = x_{-i_1}(r_1) \dots x_{-i_q}(r_q)$$

for some uniquely defined $r_1(a), \dots, r_q(a) > 0$ (see [2]). Let $u_k(a) = r_k(a) e^{-\alpha_{i_k}(a(T))}$.

Theorem 6.5. *Let $(x_1, \dots, x_q)$ be the string parametrization of a in $L_{\mathcal{P}_{w_0} a}$, associated with the decomposition* (6.8)*, then*

$$(x_1, \dots, x_q) = \lim_{\varepsilon \to 0} \varepsilon \big(\log u_1(a/\varepsilon), \dots, \log u_q(a/\varepsilon) \big).$$

Proof. When we multiply b on the right by $\bar{s}_{i_1}$, and take its Gauss decomposition

$$[b s_{i_1}]_- [b s_{i_1}]_0 [b s_{i_1}]_+ = [b]_0 [b]_+ s_{i_1}$$

then

$$[b]_+ s_{i_1} [b s_{i_1}]_+^{-1} = [b]_0^{-1} [b s_{i_1}]_- [b s_{i_1}]_0 \in N s_{i_1} N \cap B_- L^{s_{i_1}, e}$$

and

$$[b]_+ s_{i_1} [b s_{i_1}]_+^{-1} = x_{-i_1}(r_1)$$

for some r_1. In fact, using our formula for Littelmann operators,

$$r_1 = e^{\alpha_1(a(T))} \int_0^T e^{-\alpha_1(a(s))}\, ds.$$

Comparing with (3.3) we see that $r_1 e^{-\alpha_1(a(T))}$ gives a geometric lifting of the first string coordinate for the Littelmann module. We can continue the process starting from $[bs_{i_1}]_+$, to get

$$[bs_{i_1}]_+ s_{i_2} [bs_{i_1} s_{i_2}]_+^{-1} = x_{-i_2}(r_2)$$

(using the fact that $[g_1 g_2]_+ = [[g_1]_+ g_2]_+$ for $g_1, g_2 \in G$) obtaining successive decompositions

$$[b]_+ s_{i_1} \dots s_{i_k} [bs_{i_1} \dots s_{i_k}]_+^{-1} = x_{-i_1}(r_1) \dots x_{-i_k}(r_k).$$

This gives the coordinates of $[[b]_+ w_0]_{-0} \in L^{w_0,e}$, which are thus seen to correspond to the string coordinates by a geometric lifting. □

Appendix A

This appendix is devoted to the proof of Theorem 2.6.

Lemma A.1. *If $B(\lambda)$, $\lambda \in \overline{C}$, is a closed normal family of highest weight continuous crystals then for each $\lambda, \mu \in \overline{C}$ such that $\lambda \leqslant \mu$ there exists an injective map $\Psi_{\lambda,\mu} : B(\lambda) \to B(\mu)$ with the following properties:*

(i) $\Psi_{\lambda,\mu}(b_\lambda) = b_\mu$,
(ii) $\Psi_{\lambda,\mu} e_\alpha^r(b) = e_\alpha^r \Psi_{\lambda,\mu}(b)$, *for all* $b \in B(\lambda)$, $\alpha \in \Sigma$, $r \geqslant 0$,
(iii) $\Psi_{\lambda,\mu} f_\alpha^r(b) = f_\alpha^r \Psi_{\lambda,\mu}(b)$ *if* $f_\alpha^r(b) \in B(\lambda)$.

Proof. Let $\nu = \mu - \lambda$. First consider the map $\phi_{\lambda,\mu} : B(\lambda) \to B(\lambda) \otimes B(\nu)$ given by $\phi_{\lambda,\mu}(b) = b \otimes b_\nu$, when $b \in B(\lambda)$. Since b_ν is a highest weight $\varepsilon_\alpha(b_\nu) = 0$. By normality, for all $b \in B(\lambda)$, $\varphi_\alpha(b) \geqslant 0$. Therefore $\sigma := \varphi_\alpha(b) - \varepsilon_\alpha(b_\nu) = \varphi_\alpha(b) \geqslant 0$. By definition, this implies that $\varepsilon_\alpha(b \otimes b_\nu) = \varepsilon_\alpha(b)$, $\varphi_\alpha(b \otimes b_\nu) = \varphi_\alpha(b)$, $wt(b \otimes b_\nu) = wt(b) + \nu$. Using (2.1) we see also that, for $r \geqslant 0$, $e_\alpha^r(b \otimes b_\nu) = e_\alpha^r b \otimes b_\nu$ and that, when $f_\alpha^r(b) \in B(\lambda)$, $r \leqslant \varphi_\alpha(b) = \sigma$ by normality, and therefore $f_\alpha^r(b \otimes b_\nu) = f_\alpha^r b \otimes b_\nu$. Since the family is closed there is an isomorphism $i_{\lambda,\mu} : \mathcal{F}(b_\lambda \otimes b_\nu) \to B(\mu)$. One has $i_{\lambda,\mu}(b_\lambda \otimes b_\nu) = b_\mu$. One can take $\Psi_{\lambda,\mu} = i_{\lambda,\mu} \circ \phi_{\lambda,\mu}$. □

The family $\Psi_{\lambda,\mu}$ constructed above satisfies $\Psi_{\lambda,\lambda} = id$ and, when $\lambda \leqslant \mu \leqslant \nu$, $\Psi_{\mu,\nu} \circ \Psi_{\lambda,\mu} = \Psi_{\lambda,\nu}$, so that we can consider the direct limit $B(\infty)$ of the family $B(\lambda)$, $\lambda \in \overline{C}$, with the injective maps $\Psi_{\lambda,\mu} : B(\lambda) \to B(\mu)$, $\lambda \leqslant \mu$. Still following Joseph [18], we define a crystal structure on $B(\infty)$.

Proposition A.2. *The direct limit $B(\infty)$ is a highest weight upper normal continuous crystal with highest weight* 0.

Proof. By definition, the direct limit $B(\infty)$ is the quotient set $B/\sim$ where $B = \bigcup_{\lambda \in \overline{C}} B(\alpha)$ is the disjoint union of the $B(\lambda)$'s and where $b_1 \sim b_2$ for $b_1 \in B(\lambda)$, $b_2 \in B(\mu)$, when there exists a $\nu \in \overline{C}$ such that $\nu \geqslant \lambda$, $\nu \geqslant \mu$ and $\Psi_{\lambda,\nu}(b_1) = \Psi_{\mu,\nu}(b_2)$. Let $\bar{b}$ be the image in $B(\infty)$ of $b \in B$. If $b \in B(\lambda)$, then we define $wt(\bar{b}) = wt(b) - \lambda$, $\varepsilon_\alpha(\bar{b}) = \varepsilon_\alpha(b)$, $\varphi_\alpha(\bar{b}) = \varepsilon_\alpha(\bar{b}) + \alpha^\vee(wt(\bar{b}))$ and, when $r \geqslant 0$, $e_\alpha^r(\bar{b}) = \overline{e_\alpha^r(b)}$. These do not depend on λ, since if $\mu \geqslant \lambda$ and $b' = \Psi_{\lambda,\mu}(b)$, then one has $\bar{b}' = \bar{b}$ and $wt(b') = wt(b) + \mu - \lambda$. In order to define $f_\alpha^r(\bar{b})$ for $r \geqslant 0$, let us choose $\mu \geqslant \lambda$ large enough to ensure that

$$\varphi_\alpha(b') = \varepsilon_\alpha(b') + \alpha^\vee\big(wt(b)\big) + \alpha^\vee(\mu - \lambda) \geqslant r.$$

Then $f_\alpha^r b' \neq \mathbf{0}$ by normality and we define $f^r \bar{b} = \overline{f^r b'}$. Again this does not depend on μ. Using the lemma we check that this defines a crystal structure on $B(\infty)$. Each $\Psi_{\lambda,\mu}$, $\lambda \leqslant \mu$, commutes with the e_α^r, $r \geqslant 0$. This implies that $B(\infty)$ is upper normal. Since each $B(\lambda)$ is a highest weight crystal, $B(\infty)$ has also this property. □

We will denote b_∞ the unique element of $B(\infty)$ of weight 0. Note that $B(\infty)$ is not lower normal. For instance,

$$\varphi_\alpha(b_\infty) = 0, \qquad f(b_\infty) \neq \mathbf{0}, \quad \text{for all } f \in \mathcal{F}. \tag{A.1}$$

For $\lambda \in \overline{C}$ we define the crystal $S(\lambda)$ as the set with a unique element $\{s_\lambda\}$ and the maps $wt(s_\lambda) = \lambda$, $\varepsilon_\alpha(s_\lambda) = -\alpha^\vee(\lambda)$, $\varphi_\alpha(s_\lambda) = 0$ and $e_\alpha^r(s_\lambda) = \mathbf{0}$ when $r \neq 0$.

Lemma A.3. *The map*

$$\Psi_\lambda : b \in B(\lambda) \mapsto \bar{b} \otimes s_\lambda \in B(\infty) \otimes S(\lambda)$$

is a crystal embedding.

Proof. Let $b \in B(\lambda)$, then

$$wt\big(\Psi_\lambda(b)\big) = wt(\bar{b} \otimes s_\lambda) = wt(\bar{b}) + wt(s_\lambda) = wt(b) - \lambda + \lambda = wt(b).$$

Let $\sigma = \varphi_\alpha(\bar{b}) - \varepsilon_\alpha(s_\lambda)$. Then $\sigma = \varphi_\alpha(b)$ since $\varepsilon_\alpha(s_\lambda) = -\alpha^\vee(\lambda)$ and $\varphi_\alpha(\bar{b}) = \varphi_\alpha(b) - \alpha^\vee(\lambda)$. Thus $\sigma \geqslant 0$ by normality of $B(\lambda)$. By the definition of the tensor product, this implies that

$$\varepsilon_\alpha\big(\Psi_\lambda(b)\big) = \varepsilon_\alpha(\bar{b} \otimes s_\lambda) = \varepsilon_\alpha(\bar{b}) = \varepsilon_\alpha(b),$$

thus $\varphi_\alpha(\Psi_\lambda(b)) = \varphi_\alpha(b)$. Furthermore, since $\sigma \geqslant 0$,

$$e_\alpha^r\big(\Psi_\lambda(b)\big) = e_\alpha^r(\bar{b} \otimes s_\lambda) = e_\alpha^{\max(r,-\sigma)}(\bar{b}) \otimes e^{\min(r,-\sigma)+\sigma} s_\lambda.$$

When $r \geqslant -\sigma$, this is equal to $e_\alpha^r(\bar{b}) \otimes s_\lambda = \Psi_\lambda(e_\alpha^r(b))$. If $r < -\sigma$ then $e_\alpha^r(\Psi_\lambda(b)) = e_\alpha^{-\sigma}(\bar{b}) \otimes e_\alpha^{r+\sigma}(s_\lambda) = \mathbf{0}$, since $e_\alpha^s(s_\lambda) = \mathbf{0}$ when $s \neq 0$, and on the other hand, $e_\alpha^r(b) = \mathbf{0}$ by normality. Thus $\Psi_\lambda(e_\alpha^r(b)) = \mathbf{0}$. □

If $f = f_{\alpha_n}^{r_n} \cdots f_{\alpha_1}^{r_1} \in \mathcal{F}$, we say that $f' \in F$ is extracted from f if $f' = f_{\alpha_n}^{r'_n} \cdots f_{\alpha_1}^{r'_1}$ with $0 \leqslant r'_k \leqslant r_k$, $k = 1, \ldots, n$. Recall the definition of $B_\alpha = \{b_\alpha(t), t \leqslant 0\}$ given in Example 2.2.

Lemma A.4. *Let $f \in \mathcal{F}$ and $\alpha \in \Sigma$, then there exists f' extracted from f and $t \geqslant 0$ such that*

$$f\big(b_\infty \otimes b_\alpha(0)\big) = f' b_\infty \otimes b_\alpha(-t).$$

Moreover if $\lambda \in \overline{C}$ is such that $\alpha^\vee(\lambda) = 0$ and $\beta^\vee(\lambda)$ large enough for all $\beta \in \Sigma - \{\alpha\}$, then for $\mu \in \overline{C}$, for the same $f' \in \mathcal{F}$ and $t \geqslant 0$,

$$f(b_\lambda \otimes b_\mu) = f' b_\lambda \otimes f_\alpha^t b_\mu.$$

Proof. The first part follows easily from the definition of the tensor product. Let $\lambda \in \overline{C}$ such that $\alpha^\vee(\lambda) = 0$, $\mu \in \overline{C}$, $\beta \in \Sigma - \{\alpha\}$ and $r \geqslant 0$. If, for some $s > 0$, one has $e_\beta^s(f_\alpha^r b_\mu) \neq \mathbf{0}$ then $wt(e_\beta^s(f_\alpha^r b_\mu)) = \mu + s\beta - r\alpha$ is in $\mu - \overline{C}$ (since μ is a highest weight). This is not possible because $\beta^\vee(s\beta - r\alpha) \geqslant s\beta^\vee(\beta) > 0$. Therefore, by normality, $\varepsilon_\beta(f_\alpha^r b_\mu) = 0$. On the other hand, for all $f = f_{\alpha_n}^{r_n} \cdots f_{\alpha_1}^{r_1} \in \mathcal{F}$,

$$\varphi_\beta(f b_\lambda) = \beta^\vee\big(wt(f b_\lambda)\big) + \varepsilon_\beta(f b_\lambda) \geqslant \beta^\vee\big(wt(f b_\lambda)\big) = \beta^\vee(\lambda) - \sum_{k=1}^n r_k \beta^\vee(\alpha_k).$$

Let $\sigma = \varphi_\beta(f b_\lambda) - \varepsilon_\beta(f_\alpha^r b_\mu) = \varphi_\beta(f b_\lambda)$ and $s \geqslant 0$. Then

$$\sigma = \varphi_\beta(f b_\lambda) \geqslant \beta^\vee(\lambda) - \sum_{k=1}^n r_k \beta^\vee(\alpha_k).$$

If $\beta^\vee(\lambda)$ is large enough, then $\sigma \geqslant \max(s, 0)$ which implies, see (2.1), that

$$f_\beta^s\big(f b_\lambda \otimes f_\alpha^r b_\mu\big) = \big(f_\beta^s f b_\lambda\big) \otimes f_\alpha^r b_\mu. \tag{A.2}$$

On the other hand, $\varphi_\alpha(b_\lambda) = \alpha^\vee(\lambda) + \varepsilon_\alpha(b_\lambda) = 0$, since $\varepsilon_\alpha(b_\lambda) = 0$ by normality. We also know that $\varphi_\alpha(b_\infty) = 0$, see (A.1), hence

$$\varphi_\alpha(f b_\lambda) = \varphi_\alpha(b_\lambda) - \sum_{k=1}^n r_k \alpha^\vee(\alpha_k) = \varphi_\alpha(b_\infty) - \sum_{k=1}^n r_k \alpha^\vee(\alpha_k) = \varphi_\alpha(f b_\infty).$$

Thus $\sigma = \varphi_\alpha(f b_\infty)$ and does not depend on λ. It follows that the following decomposition is independent of λ:

$$f_\alpha^s\big(f b_\lambda \otimes f_\alpha^r b_\mu\big) = f_\alpha^{\sigma \wedge s} f b_\lambda \otimes f_\alpha^{r+s-\sigma \wedge s} b_\mu. \tag{A.3}$$

Using (A.2) and (A.3), it is now easy to prove the lemma by induction on n, proving first the second assertion. □

Proposition A.5. *For each simple root α, there is a crystal embedding $\Gamma_\alpha : B(\infty) \to B(\infty) \otimes B_\alpha$ such that $\Gamma_\alpha(b_\infty) = b_\infty \otimes b_\alpha(0)$.*

Proof. Let us show that the expression

$$\Gamma_\alpha(fb_\infty) = f\big(b_\infty \otimes b_\alpha(0)\big), \quad f \in \mathcal{F}, \tag{A.4}$$

defines the morphism Γ_α. First we check that it is well defined. By definition, $fb_\infty = \overline{fb_\nu}$ for all $\nu \in \overline{C}$ such that $\overline{fb_\nu} \neq \mathbf{0}$.

Let us choose λ as in Lemma A.4. For $\mu \in \overline{C}$ large enough, $\overline{fb_{\lambda+\mu}} \neq \mathbf{0}$. Let us write

$$\overline{fb_{\lambda+\mu}} = f(\bar{b}_\lambda \otimes \bar{b}_\mu) = \overline{f'b_\lambda} \otimes \overline{f_\alpha^t b_\mu}.$$

Then f' and t depend only on $fb_{\lambda+\mu}$, which by definition depends only on fb_∞. By Lemma A.4,

$$f\big(b_\infty \otimes b_\alpha(0)\big) = f'b_\infty \otimes b_\alpha(-t)$$

which depends only on fb_∞ (and not on f itself), showing that Γ_α is well defined on $\mathcal{F}b_\infty$, and thus on $B(\infty)$, since $\mathcal{F}b_\infty = B(\infty)$. Notice that $f(b_\infty \otimes b_\alpha(0)) \neq \mathbf{0}$ since $f'b_\infty \neq \mathbf{0}$.

Let us prove that Γ_α is injective. Suppose that $f(b_\infty \otimes b_\alpha(0)) = \tilde{f}(b_\infty \otimes b_\alpha(0))$ for some $f, \tilde{f} \in \mathcal{F}$. Using Lemma A.4,

$$f\big(b_\infty \otimes b_\alpha(0)\big) = f'b_\infty \otimes b_\alpha(-t) \quad \text{and} \quad \tilde{f}\big(b_\infty \otimes b_\alpha(0)\big) = \tilde{f}'b_\infty \otimes b_\alpha(-\tilde{t}).$$

If $\lambda \in \overline{C}$ is as in this lemma, then

$$f(b_\lambda \otimes b_\mu) = f'b_\lambda \otimes f_\alpha^t(b_\mu) = \tilde{f}'b_\lambda \otimes f_\alpha^{\tilde{t}} b_\mu = \tilde{f}(b_\lambda \otimes b_\mu),$$

therefore $fb_{\lambda+\mu} = \tilde{f}b_{\lambda+\mu}$, thus $fb_\infty = \tilde{f}b_\infty$. It is clear that Γ_α commutes with f_α^r, $r \geqslant 0$. Since $\varepsilon_\alpha(b_\alpha(0)) = \varphi_\alpha(b_\infty) = 0$,

$$\varepsilon_\alpha\big(\Gamma_\alpha(b_\infty)\big) = \varepsilon_\alpha\big(b_\infty \otimes b_\alpha(0)\big) = \varepsilon_\alpha(b_\infty),$$

hence, if $f = f_{\alpha_n}^{r_n} \cdots f_{\alpha_1}^{r_1} \in \mathcal{F}$,

$$\varepsilon_\alpha\big(\Gamma_\alpha(fb_\infty)\big) = \varepsilon_\alpha\big(f\Gamma_\alpha(b_\infty)\big) = \varepsilon_\alpha\big(\Gamma_\alpha(b_\infty)\big) - \sum_{k=1}^{n} r_k \beta^\vee(\alpha_k) = \varepsilon_\alpha(fb_\infty).$$

Therefore Γ_α commutes with ε_α. It also commutes with wt since $wt(b_\infty) = 0$. Let us now consider e_α^r, $r \geqslant 0$. Let $b \in B(\infty)$. If $e_\alpha^r(b) \neq \mathbf{0}$, then

$$\Gamma_\alpha(b) = \Gamma_\alpha\big(f_\alpha^r e_\alpha^r(b)\big) = f_\alpha^r\big(\Gamma_\alpha\big(e_\alpha^r(b)\big)\big) \neq \mathbf{0}$$

hence $\Gamma_\alpha(e_\alpha^r(b)) = e_\alpha^r(\Gamma_\alpha(b))$. Suppose now that $e_\alpha^r(b) = \mathbf{0}$. Since $B(\infty)$ is upper normal, one has $\varepsilon_\alpha(b) = 0$, hence $\varepsilon_\alpha(\Gamma_\alpha(b)) = 0$. By the lemma, there is $f' \in \mathcal{F}$ and $t \geqslant 0$ such that $\Gamma_\alpha(b) = \Gamma_\alpha(b) = f'b_\infty \otimes b_\alpha(-t)$. Therefore

$$0 = \varepsilon_\alpha\big(\Gamma_\alpha(b)\big) \geqslant \varepsilon_\alpha(f'b_\infty) \geqslant 0.$$

By upper normality this implies that $e_\alpha^r(f'b_\infty)=\mathbf{0}$, hence

$$e_\alpha^r\big(\Gamma_\alpha(b)\big)=e_\alpha^r\big(f'b_\infty\otimes b_\alpha(-t)\big)=\big(e_\alpha^r f'b_\infty\big)\otimes b_\alpha(-t)=\mathbf{0}. \qquad \square$$

The following lemma is clear.

Lemma A.6. *Let B_1, B_2 and C be three continuous crystals and $\psi:B_1\to B_2$ be crystal embeddings. Then $\tilde\psi:B_1\otimes C\to B_2\otimes C$ defined by $\tilde\psi(b\otimes c)=\psi(b)\otimes c$ is a crystal embedding.*

A.1. Uniqueness. Proof of Theorem 2.6

Recall that Σ is the set of simple roots. Fix a sequence $A=(\dots,\alpha_2,\alpha_1)$ of elements of Σ such that each simple root occurs infinitely many times and $\alpha_n\neq\alpha_{n+1}$ for all $n\geqslant 1$. Let $\hat B(A)$ be the subset of $\cdots\otimes B_{\alpha_2}\otimes B_{\alpha_1}$ in which the kth entry differs from $b_{\alpha_k}(0)$ for only finitely many k. One checks that the rules given for the multiple tensor give $\hat B(A)$ the structure of a continuous crystal (see, e.g., Kashiwara [21, 7.2], Joseph [17,18]). Let b_A be the element of $\hat B(A)$ with entries $b_{\alpha_n}(0)$ for all $n\geqslant 1$. We denote $B(A)=\mathcal{F}b_A$.

Proposition A.7. *There exists a crystal embedding Γ from $B(\infty)$ onto $B(A)$ such that $\Gamma(b_\infty)=b_A$.*

Proof. Let $f\in\mathcal{F}$. We can write $f=f_{\alpha_k}^{r_k}\cdots f_{\alpha_1}^{r_1}$ where $(\dots,\alpha_2,\alpha_1)=A$ and $r_n\geqslant 0$ for all $n\geqslant 1$. By Lemma A.4

$$\Gamma_{\alpha_1}\big(f_{\alpha_1}^{r_1}(b_\infty)\big)=f_{\alpha_1}^{r_1}(\Gamma_{\alpha_1}b_\infty)=f_{\alpha_1}^{r_1}\big(b_\infty\otimes b_{\alpha_1}(0)\big)=b_\infty\otimes b_{\alpha_1}(-r_1),$$

therefore

$$\Gamma_{\alpha_1}\big(f_{\alpha_k}^{r_k}\cdots f_{\alpha_1}^{r_1}b_\infty\big)=\big(f_{\alpha_k}^{r_k'}\cdots f_{\alpha_2}^{r_2'}b_\infty\big)\otimes b_{\alpha_1}\big(-r_1'\big)$$

for some $r_1',\dots,r_k'\geqslant 0$. Similarly,

$$\Gamma_{\alpha_2}\big(f_{\alpha_k}^{r_k'}\cdots f_{\alpha_2}^{r_2'}b_\infty\big)=\big(f_{\alpha_k}^{r_k''}\cdots f_{\alpha_3}^{r_3''}b_\infty\big)\otimes b_{\alpha_2}\big(-r_2''\big)$$

for some $r_2'',r_3'',\dots,r_k''$. If we apply Lemma A.6 to $B_1=B(\infty)$, $B_2=B(\infty)\otimes B_{\alpha_2}$, $\psi=\Gamma_{\alpha_2}$, $C=B_{\alpha_1}$, we obtain a crystal embedding

$$\tilde\Gamma_{\alpha_2}:B(\infty)\otimes B_{\alpha_1}\to B(\infty)\otimes B_{\alpha_2}\otimes B_{\alpha_1}$$

such that, for $b\in B(\infty)$, $b_1\in B_{\alpha_1}$

$$\tilde\Gamma_{\alpha_2}(b\otimes b_1)=\Gamma_{\alpha_2}b\otimes b_1.$$

Let $\Gamma_{\alpha_2,\alpha_1}=\tilde\Gamma_{\alpha_2}\circ\Gamma_{\alpha_1}:B(\infty)\to B(\infty)\otimes B_{\alpha_2}\otimes B_{\alpha_1}$, then

$$\begin{aligned}\Gamma_{\alpha_2,\alpha_1}\big(f_{\alpha_k}^{r_k}\cdots f_{\alpha_1}^{r_1}b_\infty\big) &= \tilde{\Gamma}_{\alpha_2}\big(f_{\alpha_k}^{r'_k}\cdots f_{\alpha_2}^{r'_2}b_\infty\otimes b_{\alpha_1}\big(-r'_1\big)\big)\\ &= \Gamma_{\alpha_2}\big(f_{\alpha_k}^{r'_k}\cdots f_{\alpha_2}^{r'_2}b_\infty\big)\otimes b_{\alpha_1}\big(-r'_1\big)\\ &= \big(f_{\alpha_k}^{r''_k}\cdots f_{\alpha_3}^{r''_3}b_\infty\big)\otimes b_{\alpha_2}\big(-r''_2\big)\otimes b_{\alpha_1}\big(-r'_1\big).\end{aligned}$$

Again, with Γ_{α_3} we build $\Gamma_{\alpha_3,\alpha_2,\alpha_1}=\tilde{\Gamma}_{\alpha_3}\circ\Gamma_{\alpha_2,\alpha_1}$. Inductively we obtain strict morphisms

$$\Gamma_{\alpha_k,\dots,\alpha_1}:B(\infty)\to B(\infty)\otimes B_{\alpha_k}\otimes\cdots\otimes B_{\alpha_2}\otimes B_{\alpha_1}$$

such that for some $s_k,\dots,s_1$

$$\Gamma_{\alpha_k,\dots,\alpha_1}\big(f_{\alpha_k}^{r_k}\cdots f_{\alpha_1}^{r_1}b_\infty\big)=b_\infty\otimes b_{\alpha_k}(-s_k)\otimes\cdots\otimes b_{\alpha_1}(-s_1).$$

Now we can define $\Gamma:B(\infty)\to B(A)$ by the formula

$$\Gamma\big(f_{\alpha_k}^{r_k}\cdots f_{\alpha_1}^{r_1}b_\infty\big)=\cdots\otimes b_{\alpha_{k+n}}(0)\otimes\cdots\otimes b_{\alpha_{k+1}}(0)\otimes b_{\alpha_k}(-s_k)\otimes\cdots\otimes b_{\alpha_1}(-s_1).$$

One checks that this is a crystal embedding. □

This shows that $B(\infty)$ is isomorphic to $B(A)$, which does not depend on the chosen closed family of crystals, and thus proves the uniqueness. It also shows that $B(A)$ does not depend on A, as soon as a closed family exists.

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