Free fermions and α-determinantal processes

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Abstract

The α-determinant is a one-parameter generalisation of the standard determinant, with $\alpha = -1$ corresponding to the determinant, and $\alpha = 1$ corresponding to the permanent. In this paper a simple limit procedure to construct α-determinantal point processes out of fermionic processes is examined. The procedure is illustrated for a model of N free fermions in a harmonic potential. When the system is in the ground state, the rescaled correlation functions converge for large N to determinants (of the sine kernel in the bulk and the Airy kernel at the edges). We analyse the point processes associated to a special family of excited states of fermions and show that appropriate scaling limits generate α-determinantal processes. Links with wave optics and other random matrix models are suggested.

Keywords: random point processes, non-interacting fermions, α-determinantal processes, random matrices

(Some figures may appear in colour only in the online journal)

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1. Introduction

Determinantal and permanental processes are point processes whose correlation functions $\varrho_n(x_1, \ldots, x_n)$ exist for all $n \in \mathbb{N}$, and are given by

$$\varrho_n(x_1, \ldots, x_n) = \begin{cases} \det_{1 \leqslant i,j \leqslant n} K(x_i, x_j) & \text{(determinantal)} \\ \operatorname*{per}_{1 \leqslant i,j \leqslant n} K(x_i, x_j) & \text{(permanental).} \end{cases} \tag{1}$$

The function $K(x, y)$ is called *correlation kernel* and can be thought of as the integral kernel of some integral operator. There is no need for us to review the history and ubiquity of determinantal and permanental processes in mathematical physics and probability [30, 35, 43, 44]. Another, perhaps not so well-known class of processes are the so-called α-determinantal processes. The α-determinant of a $n \times n$ matrix A is

$$\det\nolimits_\alpha A = \sum_{\sigma \in S_n} \alpha^{n-m(\sigma)} A_{\sigma(1)1} A_{\sigma(2)2} \cdots A_{\sigma(n)n} \tag{2}$$

where $m(\sigma)$ is the number of disjoint cycles in the permutation σ — thus, for example, the identity permutation, corresponding to the term $A_{11}A_{22} \cdots A_{nn}$ contains n cycles and appears with weight α^0, whereas the term $A_{12}A_{23} \cdots A_{n1}$, corresponding to a single cycle appears with weight α^{n-1}. Namely, we simply replace the signature $\operatorname{sgn}(\sigma) = (-1)^{n-m(\sigma)}$ by $\alpha^{n-m(\sigma)}$ in the definition of the ordinary determinant $\det A$.

It is clear that

$$\det\nolimits_{-1} A = \det A, \quad \det\nolimits_1 A = \operatorname{per} A, \quad \det\nolimits_0 A = A_{11}A_{22} \cdots A_{nn}. \tag{3}$$

Vere-Jones [45, 46] introduced α-determinants to treat the probability density functions of multivariate binomial and negative binomial distributions in a unified way. Later, Shirai and Takahashi [41] utilised the α-determinant to define a parametric family of point processes which extend the fermionic and bosonic point processes. Let $\alpha \in \mathbb{R}$ and K a kernel from say $\mathbb{R}^2$ to $\mathbb{C}$. An α-determinantal point process with kernel K is defined, when it exists, as the point process with n-point correlation functions ($n \geqslant 1$)

$$\varrho_n(x_1, \ldots, x_n) = \det\nolimits_\alpha_{1 \leqslant i,j \leqslant n} K(x_i, x_j). \tag{4}$$

The values $\alpha = -1$ and $\alpha = 1$ correspond to determinantal and permanental processes, respectively. The case $\alpha = 0$ corresponds to the Poisson process with intensity $K(x, x)$.

Several authors have established necessary and sufficient conditions for the existence of α-determinantal processes. See [34] and references therein.

In this paper, we shall only be concerned with the case $\alpha < 0$; in this case, a necessary condition for existence is that is that $-\frac{1}{\alpha} \in \mathbb{N}$ (otherwise the α-determinants $\det_\alpha K(x_i, x_j)$ can be negative). If $-\frac{1}{\alpha} \in \mathbb{N}$, and K is self-adjoint with $0 \leqslant K \leqslant -\frac{1}{\alpha}$, then the α-determinantal process exists. In fact, it is just a union (or 'superposition') of $-\frac{1}{\alpha}$ i.i.d. copies of the determinantal process with kernel $-\alpha K$.

Although α-determinantal processes have been investigated theoretically, concrete realisations of them have not been discussed as much in the literature. The present paper might be thought of as a first step in this direction; hopefully more examples will emerge in time.

The purpose of this paper is to provide an explicit construction of α-determinantal point processes as limiting cases arising naturally in a model of N non-interacting fermions in a one-dimensional harmonic potential. We consider a family of many-body excited states parametrized by a real number a, where $a = 0$ corresponds to the fermionic ground state. The

associated determinantal process is a *block projection process*. The first observation of the paper is that, as the parameter varies from $a = 0$ to $a \to \infty$, the average density of fermions crosses over from the Wigner semicircular distribution (in the quantum ground state) to the arcsine distribution (corresponding to a fully 'classical' excited state); this is consistent with the correspondence principle of quantum mechanics. The main result of the paper is that if the limit $a \to \infty$ is taken appropriately, then the block projection process associated to the many-body excited state converges weakly (in the scaling limit) to an α-determinantal process with $\alpha = -1/2$. In the same setting, we also provide the explicit construction of α-determinantal processes for general $\alpha = -1/m$, with $m \in \mathbb{N}$. These results are summarised as a theorem in section 5.

The outline of the paper is as follows. In the next section we record the spectral properties of non-interacting fermions in a harmonic potential. In section 3, we first recall the connection between free fermions in the ground state and the GUE processes, and some immediate implications of this connection; then, we introduce a first example of block projection process and we analyse its scaling limits and the convergence to an α-determinantal process. In section 4 we generalise the construction of block projection processes and show their convergence to α-determinantal processes (superposition of sine processes). A summary of the main result— weak convergence of block projection fermionic processes to α-determinantal processes, further remarks and links with wave optics and random matrices conclude the paper (section 5).

Some notation. For $a < b$ we use the notation $[a \mathinner{.\,.} b)$ to denote the integer interval $\{\lfloor a \rfloor, \lfloor a \rfloor + 1, \ldots, \lfloor b \rfloor - 1\}$. For $x_i, x_j \in \mathbb{R}$ we write $x_{ij} = x_i - x_j$. Denote the complex conjugate of z by $\bar{z}$.

2. Free fermions in a harmonic potential and determinantal processes

The connection between free fermions and determinantal processes has been known for a long time [15, 30, 31, 35, 43]. This connection has been used in various contexts, such as in the analysis of a class of matrix models (Moshe–Neuberger–Shapiro model) [22, 36], in the study of non-intersecting step-edges on a crystal [11], and in establishing a connection between non-intersecting Brownian interfaces in the presence of a confining potential and Wishart random matrices [37]. However, in the specific context of N non-interacting fermions trapped in a one-dimensional harmonic potential, the connection to the Gaussian unitary ensemble (GUE) was established and used only recently in a series of papers: first somewhat indirectly in [12, 47], and then more explicitly in [13, 32] in the context of full counting statistics of fermions. Later, this connection has been further exploited quite heavily in calculating various physical properties of 1-d trapped fermions, such as the correlation functions near the edges of the trapped Fermi gas [6, 7, 27], effects of finite temperature and the connection to the Kardar–Parisi–Zhang equation at finite time [6, 7], computation of the number variance, other linear statistics [20, 21, 32, 33] and the entanglement entropy [4]. Free fermions in a one-dimensional non-harmonic traps, singular or with hard edges such as a box potential (where the determinantal process is not GUE), have also been studied [23, 24]. In particular, the relationship between fermions in a box with different boundary conditions and the classical compact groups have been explored [5, 16]. For a review of some of these recent developements in the physics literature, see [9]. In this section, we first recall the precise connection between the *ground state* of non-interacting fermions in a harmonic potential and the GUE determinantal process and then extend this to a class of *special excited* states that, in a certain appropriate limit of high energy, converges to α-determinantal process with $\alpha = -1/2$.

Denote by $\psi_k(x)$ the Hermite wavefunctions

$$\psi_k(x) = h_k(x)\mathrm{e}^{-x^2/4}, \qquad h_k(x) = \frac{(-1)^k}{\sqrt{\sqrt{2\pi}k!}}\,\mathrm{e}^{x^2/2}\frac{\partial^k}{\partial x^k}\mathrm{e}^{-x^2/2}, \qquad k = 0, 1, 2, \ldots. \tag{5}$$

They are solutions of the Schrödinger equation

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{x^2}{4}\right)\psi_k(x) = \left(k + \frac{1}{2}\right)\psi_k(x), \quad x \in \mathbb{R}, \tag{6}$$

and form a complete orthonormal system in $L^2(\mathbb{R}; \mathrm{d}x)$. Physically, $\psi_k(x)$ is an eigenfunction of the quantum harmonic oscillator corresponding to the eigenvalue $E_k = k + 1/2$, i.e. the eigenstate of a quantum particle in a harmonic potential at the energy level E_k.

The normalised eigenstates $\Psi_{k_1,\ldots,k_N}(x_1, \ldots, x_N)$ of a system of N spin-polarized fermions in the same harmonic potential, are given by antisymmetric linear combinations of the ψ_k's, and can be conveniently written as Slater determinants

$$\Psi_{k_1,\ldots,k_N}(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}}\det_{1\leqslant i,j\leqslant N}\psi_{k_i}(x_j), \qquad \text{with } 0 \leqslant k_1 < k_2 < \cdots < k_N. \tag{7}$$

They are eigenfunctions of the operator $\sum_i\left(-\frac{\partial^2}{\partial x_i^2} + \frac{x_i^2}{4}\right)$ with eigenvalues $E = k_1 + \cdots + k_N + N/2$, in the subspace of completely antisymmetric states $\Psi(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) = \mathrm{sgn}(\sigma)\Psi_{k_1,\ldots,k_N}(x_1, \ldots, x_N)$. These facts follow from the basic properties of the determinant.

The modulus square of the wave function $\Psi(x_1, \ldots, x_N)$ can be interpreted as the joint probability density of the particles positions. If we denote $J = \{k_1, \ldots, k_N\} \subset \mathbb{N}$, we can write

$$|\Psi_J(x_1, \ldots, x_N)|^2 = \frac{1}{N!}\det_{1\leqslant i,j\leqslant N}K_J(x_i, x_j), \tag{8}$$

where

$$K_J(x, y) = \sum_{k\in J}\overline{\psi}_k(x)\psi_k(y) \tag{9}$$

is the integral kernel of the projection operator onto the N-dimensional subspace span $\{\psi_k(x)\colon k \in J\} \subset L^2(\mathbb{R}; \mathrm{d}x)$. In fact, $|\Psi_J(x_1, \ldots, x_N)|^2$ defines a determinantal point process of N particles on $\mathbb{R}$ with respect to $\mathrm{d}x$ with kernel $K_J(x,y)$. The nth correlation function of the process is

$$\varrho_n(x_1, \ldots, x_n) = \frac{N!}{(N-n)!}\int|\Psi_J(x_1, \ldots, x_N)|^2\mathrm{d}x_{n+1}\cdots\mathrm{d}x_N = \det_{1\leqslant i,j\leqslant n}K_J(x_i, x_j). \tag{10}$$

The main observation of this paper is that, for special choices of the energy levels $J \subset \mathbb{N}$, the scaling limit in the bulk of the determinantal process defined by $K_J(x, y)$ is an α-determinantal process.

3. Free fermions

3.1. Ground state and the GUE eigenvalue process

Suppose that $J = [0 . . N)$ corresponding to the wavefunction

$$\Psi_J(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det_{1 \leqslant i,j \leqslant N} \psi_{i-1}(x_j). \tag{11}$$

This is the unique ground state of N non-interacting fermions in a harmonic potential (exactly one fermion in each energy state $k_i = i - 1$, $i = 1, \ldots, N$). The ground state energy is

$$E_0 = \sum_{k \in [0..N)} (k + 1/2) = \frac{N^2}{2}. \tag{12}$$

The kernel

$$K_J(x, y) = \sum_{k \in [0..N)} \overline{\psi}_k(x) \psi_k(y) \tag{13}$$

coincides with the kernel of the GUE ensemble of random matrix theory. The determinantal point process on $\mathbb{R}$ defined by $K_J(x, y)$ above is know as *GUE process*.

At first, a large N asymptotics of (13) seems hopeless since the number of terms in the sum is N. However, for the special choice $J = [0..N)$, one can apply the Christoffel–Darboux formula and rewrite the kernel in the form

$$K_J(x, y) = \sqrt{N} \, \frac{\psi_N(x)\psi_{N-1}(y) - \psi_{N-1}(x)\psi_N(y)}{x - y}, \tag{14}$$

which is amenable of a large N analysis by means of the Plancherel–Rotach asymptotic expansions of Hermite polynomials.

It is well-known, for instance, that the number density of particles (one-point function) is asymptotic to the semicircular law at leading order in N

$$\varrho_1(x) = K_J(x, x) \overset{N \to \infty}{\sim} \frac{1}{2\pi} \sqrt{(4N - x^2)_+}, \tag{15}$$

with the normalization $\int \varrho_1(x) \mathrm{d}x = N$. Moreover, in the scaling limit in the bulk, the GUE process converges to the sine process, a determinantal process on $\mathbb{R}$ with translation invariant kernel

$$\lim_{N \to \infty} \frac{1}{\varrho_1(0)} K_J \left(\frac{x}{\varrho_1(0)}, \frac{y}{\varrho_1(0)} \right) = \frac{\sin \pi(x - y)}{\pi(x - y)} = \frac{\sin \pi x \cos \pi y - \cos \pi x \sin \pi y}{\pi(x - y)}. \tag{16}$$

The behaviour of the process at the endpoints $\pm\sqrt{4N}$ of the density is different. At the edges, on the scale $\mathrm{O}(N^{-1/6})$ of the typical distance between points, the process converges to the Airy process with kernel

$$\lim_{N \to \infty} \frac{1}{N^{\frac{1}{6}}} K_J \left(\sqrt{4N} + \frac{x}{N^{\frac{1}{6}}}, \sqrt{4N} + \frac{y}{N^{\frac{1}{6}}} \right) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x - y}. \tag{17}$$

3.2. Excited states, the correspondence principle and α-determinants

Consider now the case $J = [a^2M..(a+1)^2M)$, labelling an excited state where N fermions occupy $N = |J|$ consecutive levels[4] $k = a^2M, \ldots, (a+1)^2M - 1$ with

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[4] We omit, for notational simplicity, to indicate explicitly the integer parts $\lfloor a^2M \rfloor, \ldots \lfloor (a+1)^2M - 1 \rfloor$; we will often do this below without repeating this warning.

Excited state
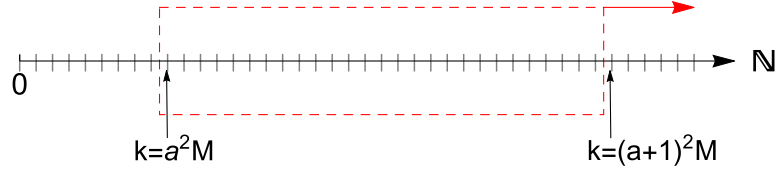
Figure 1. An excited state $J = \left[a^2M \mathinner{.\,.} (a+1)^2M\right)$ consisting of $N = ((a+1)^2 - a^2)M = (2a+1)M$ consecutive energy levels $(k+1/2)$ of the harmonic oscillator, starting with $k = a^2M$ and ending with $k = (a+1)^2M - 1$. The rectangle denoting the block J is parametrised by a, with $a = 0$ corresponding to the ground state of N fermions (when $J = [0 \mathinner{.\,.} N)$). By increasing a, one can slide the block and consider a family of such J's labelled by the single parameter a.

$N = ((a+1)^2 - a^2)M = (2a+1)M$. Thus this excited state (or 'block' as shown by the rectangle in figure 1) is parametrised by a, with $a = 0$ corresponding to the ground state. The fermions in this block J forms a determinantal process with kernel

$$K_J(x,y) = \sum_{k=a^2M}^{(a+1)^2M-1} \overline{\psi_k}(x)\psi_k(y). \tag{18}$$

Note that this particular way of parametrising the block J (with the starting level $k = a^2M = a^2N/(2a+1)$) turns out to be useful to express the scaled kernel in a nice and simple way, as is shown later.

We remark that K_J can be written as a (signed) sum of two blocks:

$$K_J(x,y) = \sum_{k \in [0 \mathinner{.\,.} (a+1)^2M)} \overline{\psi_k}(x)\psi_k(y) - \sum_{k \in [0 \mathinner{.\,.} a^2M)} \overline{\psi_k}(x)\psi_k(y). \tag{19}$$

This simple observation allows to apply the Christoffel–Darboux formula to both blocks separately, and will be crucial for the following asymptotic analysis.

3.2.1. One-point function. From (19) we can understand easily that the large-M asymptotics of the one-point function (normalised to the number of particles $N = (2a+1)M$) is

$$\varrho_1(x) = K_J(x,x) \overset{M \to \infty}{\sim} \frac{1}{2\pi}\left(\sqrt{(4(a+1)^2M - x^2)_+} - \sqrt{(4a^2M - x^2)_+}\right). \tag{20}$$

For $a = 0$, so that $N = M$, this reduces to the Wigner semicircular law between $-2\sqrt{M}$ and $2\sqrt{M}$. In general, the one-point function is concentrated between the edges $\pm 2(a+1)\sqrt{M}$. Note that $\lim_{M \to \infty} \varrho_1(0) = 1/\pi$. The density for a few values of $a > 0$ is plotted in figure 2.

For large a the one-point function approaches the arcsine law

$$\varrho_1(x) \overset{a \to \infty}{\sim} \frac{1}{\pi} \frac{(2a+1)M}{\sqrt{4a^2M - x^2}} 1_{|x| < 2a\sqrt{M}}, \tag{21}$$

which is normalized to $\int \varrho_1(x)\,\mathrm{d}x = (2a+1)M = N$ over its support $x \in \left[-2a\sqrt{M}, 2a\sqrt{M}\right]$. The name 'arcsine' comes from the fact that the cumulative number density has the form
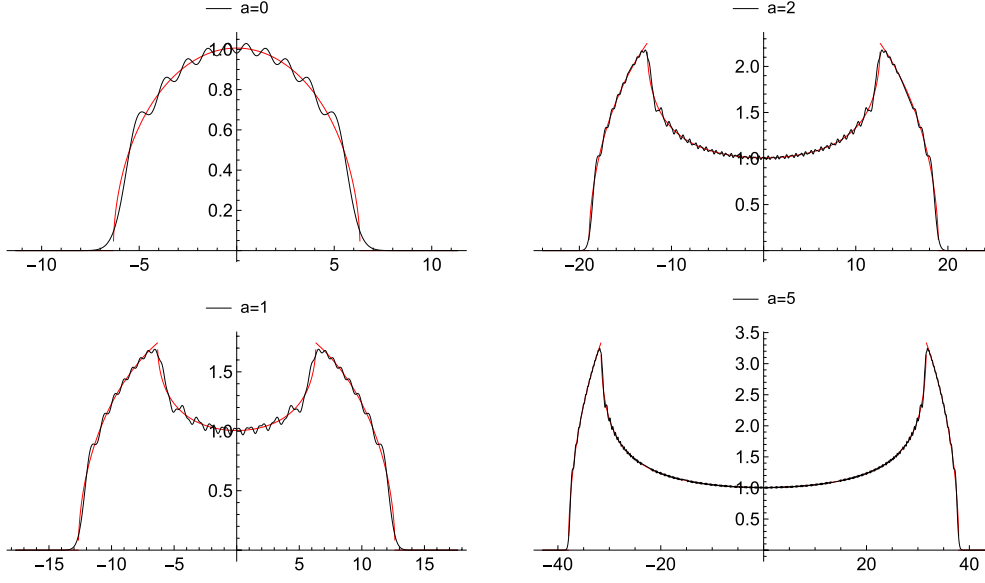
Figure 2. Comparison of the one-point functions and the $M \to \infty$ asymptotics (20). As a increases the density of states approaches the arcsine law (21).

$$\int_{-\infty}^{x} \varrho_1(x')\,\mathrm{d}x' \sim \frac{(2a+1)M}{\pi}\left[\frac{\pi}{2} + \arcsin\left(\frac{x}{2a\sqrt{M}}\right)\right]. \tag{22}$$

A semiclassical explanation for this arcsine law is as follows. The quantum state of a particle can be represented in the phase space by a quasi-probability density known as *Wigner function* (see [14]). The Wigner function $W_J(x,p)$ associated with the many-body state $\Psi_J(x_1, x_2, \ldots, x_N)$ is

$$W_J(x,p) = \frac{N}{2\pi}\int_{\mathbb{R}^N} \Psi_J^*\left(x + \frac{y}{2}, x_2, \ldots, x_N\right)\Psi_J\left(x + \frac{y}{2}, x_2, \ldots, x_N\right)\mathrm{e}^{\mathrm{i}py}\mathrm{d}y\mathrm{d}x_2\mathrm{d}x_3\ldots\mathrm{d}x_N. \tag{23}$$

For large N, the Wigner function $W_J(x,p)$ in the phase space is constant in the classically allowed region and zero in the classically forbidden region. The classically allowed region of the phase space is the set of momenta p and positions x such that the energy $E(x,p) = p^2 + x^2/4$ is between the lowest occupied energy level $E_{\min} = a^2M + 1/2$ and the largest occupied level $E_{\max} = (a+1)^2M - 1/2$.

Neglecting $o(M)$ terms the region $E_{\min} \leqslant E(x,p) \leqslant E_{\max}$ is the annulus

$$a^2M \leqslant p^2 + \frac{x^2}{4} \leqslant (a+1)^2M. \tag{24}$$

Therefore, for large $|J|$, the Wigner function (normalised to the total number of particles) is proportional to the indicator function [1, 2, 8]

$$W_J(x,p)\mathrm{d}x\mathrm{d}p \overset{M\to\infty}{\sim} \frac{1}{\pi}\mathbb{1}_{a^2M<p^2+\frac{x^2}{4}<(a+1)^2M}\mathrm{d}x\mathrm{d}p. \tag{25}$$

The projection of the Wigner function on the x-axis gives the average number density: $\varrho_1(x) = \int W_J(x,p)\mathrm{d}p$. When $a = 0$, $W_J(x,p)$ is uniform in the ellipse $p^2 + \frac{x^2}{4} \leqslant M$ (a disk if

we rescale the axes). The projection of the uniform distribution on the *disk* is the semicircular law. For $a > 1$, $W_J(x, p)$ is uniform in the annulus of radii $a\sqrt{M}$ and $(a+1)\sqrt{M}$, thus explaining the plots in figure 2. For large a, the Wigner function $W_J(x, p)$ concentrates on the circle of radius $a\sqrt{M}$, and the projection of the uniform distribution on the *circle* is the arcsine law. We will elaborate more on this key remark in the last section of the paper.

3.2.2. Scaling limits. The scaling limit in the bulk is

$$\lim_{M\to\infty} \frac{1}{\varrho_1(0)} K_J\left(\frac{x}{\varrho_1(0)}, \frac{y}{\varrho_1(0)}\right) = k(x-y), \tag{26}$$

with

$$k(x-y) = \frac{1}{\pi(x-y)} \sin\left(\pi(a+1)(x-y)\right) - \frac{1}{\pi(x-y)} \sin\left(\pi a(x-y)\right). \tag{27}$$

Using the trigonometric identity $\sin x - \sin y = 2 \sin\frac{x-y}{2} \cos\frac{x+y}{2}$, the above formula can be rearranged as

$$k(x-y) = \frac{\sin\frac{\pi}{2}(x-y)}{\frac{\pi}{2}(x-y)} \cos\omega(x-y) \tag{28}$$

where we set $\omega = \pi(a+1/2)$. For $a = 0$ this is, of course, the sine kernel.

We will now show that, as $a \to \infty$, the process becomes α-determinantal with correlation kernel $\sin\frac{\pi}{2}(x-y)/(\frac{\pi}{2}(x-y))$ and $\alpha = -1/2$.

First, we remark that for large a, the frequency ω of the cosine factor increases and $k(x-y)$ becomes rapidly oscillating. To get some insight, it is useful to write down explicitly the correlation functions

$$\tilde{\varrho}_n(x_1,\ldots,x_n) = \lim_{M\to\infty} \frac{1}{\varrho_1(0)^n} \varrho_n\left(\frac{x_1}{\varrho_1(0)},\ldots,\frac{x_n}{\varrho_1(0)}\right) = \det_{1\leqslant i,j\leqslant n} k(x_i - x_j) \tag{29}$$

for the first values of n. For all $a \geqslant 0$ the one-point function is, of course, constant

$$\tilde{\varrho}_1(x) = 1. \tag{30}$$

The two-point correlation function is

$$\tilde{\varrho}_2(x_1, x_2) = 1 - \left(\frac{\sin\frac{\pi}{2}x_{12}}{\frac{\pi}{2}x_{12}}\right)^2 \cos^2\omega x_{12}. \tag{31}$$

See figure 3 for a plot. For large ω, the factor $\cos^2\omega x_{12} = (1 + \cos\omega x_{12})/2$ rapidly oscillates around the mean value $1/2$, so

$$\lim_{\omega\to\infty} \tilde{\varrho}_2(x_1, x_2) = 1 - \frac{1}{2}\left(\frac{\sin\frac{\pi}{2}x_{12}}{\frac{\pi}{2}x_{12}}\right)^2, \tag{32}$$

where this limit is to be understood in the weak sense of integration over compact sets. In the following, all limits of correlation functions are to be understood in this sense.
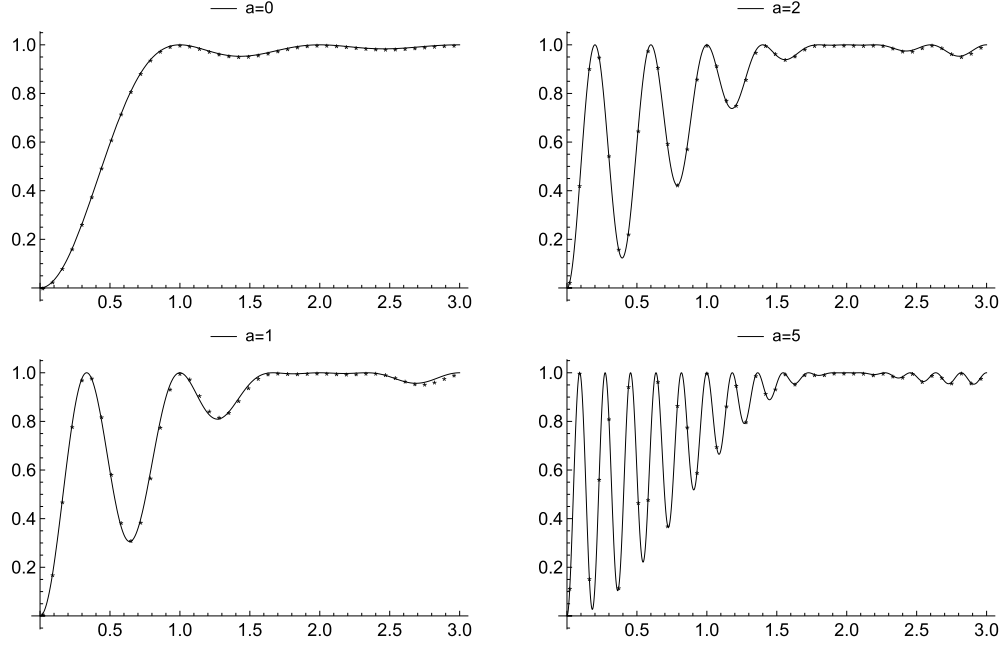
Figure 3. Scaling limit of two-point correlation functions $\varrho_1(0)^{-1}\varrho_2(\varrho_1(0)^{-1}x,$ $\varrho_1(0)^{-1}y)$, as a function of $|x-y|$, for several values of a (dots). Here $M=20$. The solid lines are the limits given in equation (32).

The correlation function for three particles is

$$\tilde{\varrho}_3(x_1,x_2,x_3) = 1 + 2\frac{\sin\frac{\pi}{2}x_{12}}{\frac{\pi}{2}x_{12}}\frac{\sin\frac{\pi}{2}x_{23}}{\frac{\pi}{2}x_{23}}\frac{\sin\frac{\pi}{2}x_{31}}{\frac{\pi}{2}x_{31}}\cos\omega x_{12}\cos\omega x_{23}\cos\omega x_{31}$$

$$- \left(\frac{\sin\frac{\pi}{2}x_{12}}{\frac{\pi}{2}x_{12}}\right)^2\cos^2\omega x_{12} - \left(\frac{\sin\frac{\pi}{2}x_{23}}{\frac{\pi}{2}x_{23}}\right)^2\cos^2\omega x_{23} - \left(\frac{\sin\frac{\pi}{2}x_{31}}{\frac{\pi}{2}x_{31}}\right)^2\cos^2\omega x_{31}.$$

$$(33)$$

Again, the squared cosines oscillate around their mean value $1/2$. The product of three cosines can be expanded as

$$\cos\omega x_{12}\cos\omega x_{23}\cos\omega x_{31}$$

$$= \cos^2\omega x_1\cos^2\omega x_2\cos^2\omega x_3 + \sin^2\omega x_1\sin^2\omega x_2\sin^2\omega x_3 + \text{zero mean terms}, \qquad (34)$$

and thus oscillates around the value $1/8 + 1/8 = 1/4$. Therefore

$$\lim_{\omega\to\infty}\tilde{\varrho}_3(x_1,x_2,x_3) = 1 - \frac{1}{2}\left(\frac{\sin\frac{\pi}{2}x_{12}}{\frac{\pi}{2}x_{12}}\right)^2 - \frac{1}{2}\left(\frac{\sin\frac{\pi}{2}x_{23}}{\frac{\pi}{2}x_{23}}\right)^2 - \frac{1}{2}\left(\frac{\sin\frac{\pi}{2}x_{31}}{\frac{\pi}{2}x_{31}}\right)^2$$

$$+ \frac{1}{2}\frac{\sin\frac{\pi}{2}x_{12}}{\frac{\pi}{2}x_{12}}\frac{\sin\frac{\pi}{2}x_{23}}{\frac{\pi}{2}x_{23}}\frac{\sin\frac{\pi}{2}x_{31}}{\frac{\pi}{2}x_{31}}, \qquad (35)$$

again in a weak sense. This pattern can be generalised for generic n as follows. The n-point correlation function is given by the determinantal formula

$$\tilde{\varrho}_n(x_1,\ldots,x_n) = \det_{1\leqslant i,j\leqslant n} k(x_i - x_j) = \sum_{\sigma\in S_n} (-1)^{n-m(\sigma)} \prod_{i=1}^{n} k(x_{\sigma(i)} - x_i)$$

$$= \sum_{\sigma\in S_n} (-1)^{n-m(\sigma)} \prod_{i=1}^{n} \frac{\sin\frac{\pi}{2}(x_i - x_{\sigma(i)})}{\frac{\pi}{2}(x_i - x_{\sigma(i)})} \prod_{i=1}^{n} \cos\omega(x_{\sigma(i)} - x_i) \tag{36}$$

($m(\sigma)$ denotes the number of cycles in $\sigma \in S_n$). For large a, the product of cosines becomes

$$\lim_{\omega\to\infty} \prod_{i=1}^{n} \cos\omega(x_{\sigma(i)} - x_i) = \left(\frac{1}{2}\right)^{n-m(\sigma)}. \tag{37}$$

We remind the reader that this limit is in the weak sense of integration over compact subsets of $\mathbb{R}^n$ or equivalently, integration against bounded measurable functions on $\mathbb{R}^n$ with compact support. For the proof, one can use the addition formulae of the trigonometric functions or, alternatively, observe that, as $\omega \to \infty$,

$$\int_0^1 \cos\omega(x - x)\mathrm{d}x = 1 \tag{38}$$

$$\int_0^1 \int_0^1 \cos^2\omega(x - y)\mathrm{d}x\mathrm{d}y = \frac{1}{2}\left(1 + \frac{\sin^2\omega}{\omega^2}\right) = \frac{1}{2} + o(1), \tag{39}$$

$$\int_0^1 \cos\omega(x - z)\cos\omega(z - y)\mathrm{d}z = \frac{1}{2}\cos\omega(x - y) + o(1). \tag{40}$$

Hence, for a given permutation $\sigma \in S_n$, we see that a cycle of length m_i contributes to the product with a factor $(1/2)^{m_i-1}$. For example, each fixed point gives a factor 1, each transposition gives a factor 1/2, a 3-cycle gives 1/4, and so on. If the $m(\sigma)$ cycles of σ have lengths m_i, $i = 1,\ldots,m(\sigma)$,

$$\prod_{i=1}^{n} \cos\omega(x_{\sigma(i)} - x_i) = \left(\frac{1}{2} + o(1)\right)^{m_1-1} \left(\frac{1}{2} + o(1)\right)^{m_2-1} \cdots \left(\frac{1}{2} + o(1)\right)^{m_{m(\sigma)}-1}$$

$$= \left(\frac{1}{2}\right)^{\sum_{i=1}^{m(\sigma)}(m_i-1)} + o(1) = \left(\frac{1}{2}\right)^{n-m(\sigma)} + o(1). \tag{41}$$

Therefore, as $a \to \infty$,

$$\tilde{\varrho}_n(x_1,\ldots,x_n) = \sum_{\sigma\in S_n} (-1)^{n-m(\sigma)} \prod_{i=1}^{n} \frac{\sin\frac{\pi}{2}(x_i - x_{\sigma(i)})}{\frac{\pi}{2}(x_i - x_{\sigma(i)})} \prod_{i=1}^{n} \cos\omega(x_{\sigma(i)} - x_i)$$

$$\overset{a\to\infty}{\to} \sum_{\sigma\in S_n} \left(-\frac{1}{2}\right)^{n-m(\sigma)} \prod_{i=1}^{n} \frac{\sin\frac{\pi}{2}(x_i - x_{\sigma(i)})}{\frac{\pi}{2}(x_i - x_{\sigma(i)})} \tag{42}$$

in the sense that

$$\lim_{a\to\infty} \int \det_{1\leqslant i,j\leqslant n} k(x_i - x_j) f(x_1,\ldots,x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n$$

$$= \int \det_{1\leqslant i,j\leqslant n} {}_{-1/2} \frac{\sin\frac{\pi}{2}(x_i - x_j)}{\frac{\pi}{2}(x_i - x_j)} f(x_1,\ldots,x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n, \tag{43}$$

for any $f(x_1, \ldots, x_n)$ bounded, measurable function with compact support. This implies convergence of gap probabilities and number density (integrated over compact sets) hence, by Kallenberg's criteria [28, theorem 4.5][29, theorem 3.3], weak convergence of the associated point processes. In particular, as $a \to \infty$, the process converges weakly to the union of two independent rescaled sine processes with kernel $\sin \frac{\pi}{2}(x - y)/(\frac{\pi}{2}(x - y))$. Note that this is not the standard sine kernel in the bulk of the ground state ($a = 0$) which is $\sin(\pi(x - y))/(\pi(x - y))$.

3.3. Local statistics at the cusps and the edges

It is clear that the previous analysis holds for any fixed point $x_0 \in \mathbb{R}$, where we have

$$\lim_{M \to \infty} \varrho_1(x_0)^{-1} K_J \left(x_0 + \varrho_1(x_0)^{-1}x, x_0 + \varrho_1(x_0)^{-1}y \right) = \frac{\sin \frac{\pi}{2}(x - y)}{\frac{\pi}{2}(x - y)} \cos \omega(x - y), \tag{44}$$

with $\omega = \frac{\pi}{2}(2a + 1)$. A look at figure 2 suggests that, for large N, the local correlations of the block projection process depend on the 'region' where we take the scaling limit. There are two points in the support of the density that look special: the cusps at $\pm 2a\sqrt{M}$ and the edges $\pm 2(a + 1)\sqrt{M}$. For example, it is clear that the local statistics at the edges cannot be described by a translation invariant kernel of the type (44). We can examine the scaling limits at points $x_0 = 2b\sqrt{M}$ not in the bulk. We report here the results (they follow from the know asymptotics (16) and (17) of the GUE process and the block structure of $K_J(x, y)$):

(i) (Before the cusp) Set $x_0 = 2b\sqrt{M}$ with $0 \leqslant b \leqslant a$:

$$\lim_{M \to \infty} \varrho_1(x_0)^{-1} K_J \left(x_0 + \varrho_1(x_0)^{-1}x, x_0 + \varrho_1(x_0)^{-1}y \right) = \frac{\sin \frac{\pi}{2}(x - y)}{\frac{\pi}{2}(x - y)} \cos \omega(x - y), \tag{45}$$

with

$$\omega = \frac{\pi}{2} \frac{\sqrt{(a + 1)^2 - b^2} + \sqrt{a^2 - b^2}}{\sqrt{(a + 1)^2 - b^2} - \sqrt{a^2 - b^2}}. \tag{46}$$

At the cusp, i.e. $b = a$, this is the sine kernel;

(ii) (After the cusp, before the edge) At $x_0 = 2b\sqrt{M}$ with $a \leqslant b < (a + 1)$:

$$\lim_{M \to \infty} \varrho_1(x_0)^{-1} K_J \left(x_0 + \varrho_1(x_0)^{-1}x, x_0 + \varrho_1(x_0)^{-1}y \right) = \frac{\sin \pi(x - y)}{\pi(x - y)}; \tag{47}$$

(iii) (At the edge) At $x_0 = 2(a + 1)\sqrt{M}$, we take the 'edge scaling' $N^{1/6}$:

$$\lim_{M \to \infty} \frac{1}{N^{1/6}} K_J \left(x_0 + \frac{x}{N^{1/6}}, x_0 + \frac{y}{N^{1/6}} \right) = \frac{\mathrm{Ai}(\eta x)\mathrm{Ai}'(\eta y) - \mathrm{Ai}'(\eta x)\mathrm{Ai}(\eta y)}{x - y}, \tag{48}$$

with $\eta = \frac{(a+1)^{1/3}}{(2a+1)^{1/6}}$. This is just a rescaling of the Airy kernel.

We remark that, when $a > 0$, between the cusps (i) the limit kernel depends explicitly on the bulk point $2b\sqrt{M}$ (as evident from (45)). This is very different from the 'quantum bulk' (ii) where the scaling limit the kernel is always the sine kernel (47) (as long as we are not at the edges). In this sense, for $a > 0$ there is a 'classical bulk' regime which is absent in the ground state $a = 0$. When $b \to a$ from below, the limit kernel freezes to the sine kernel

and no longer depends on b. It is worth noticing that this transition from (45) to (47) across the cusp $x_0 = 2a\sqrt{M}$ is continuous. We can also discuss the question of the matching in the limit of large a. When $a \to \infty$, if $b^2 = a^2 - 2\tau a$ with $\tau \geqslant 0$, then $\omega \to c\pi/2$, where $c = (1 + \sqrt{\tau/(1 + \tau)})/(1 - \sqrt{\tau/(1 + \tau)})$. If $\tau = 0$ (i.e. $b = a$) this gives the sine kernel; $\tau \to \infty$ gives $\omega \to \infty$ and we have the $-1/2$-determinantal process discussed in the previous section.

So there is a family of kernels in between with a fixed $\pi/2 < \omega < \infty$, which are seen just inside the cusp when $a \to \infty$; they are the same as if one is looking inside the bulk, between the cusps, and keeping a fixed.

4. Block projection processes

Let us summarise the limit theorems of the previous two sections in a slightly generalised setting. Consider the determinantal process with kernel (block projection)

$$K_J(x, y) = \sum_{k \in J} \overline{\psi_k}(x)\psi_k(y). \tag{49}$$

Suppose that the set of energy levels is $J = \left[a^2 M \mathinner{.\,.} (a + r)^2 M\right)$, with r positive integer. There are two cases for the rescaled processes in the bulk:

- If $a = 0$, then

$$\lim_{M \to \infty} \varrho_1(0)^{-n} \det_{1 \leqslant i,j \leqslant n} K_J(\varrho_1(0)^{-1}x_i, \varrho_1(0)^{-1}x_j) = \det_{1 \leqslant i,j \leqslant n} \frac{\sin \pi(x_i - x_j)}{\pi(x_i - x_j)}; \tag{50}$$

- If $a > 0$, then

$$\lim_{a \to \infty} \lim_{M \to \infty} \varrho_1(0)^{-n} \det_{1 \leqslant i,j \leqslant n} K_J(\varrho_1(0)^{-1}x_i, \varrho_1(0)^{-1}x_j) = \det_{-\frac{1}{2}} \frac{\sin \frac{\pi}{2}(x_i - x_j)}{\frac{\pi}{2}(x_i - x_j)} \tag{51}$$

In this section we set to ourselves to find a suitable limit procedure to obtain α-determinantal processes out of $K_J(x, y)$ with $\alpha = -\frac{1}{m}$, with m generic positive integer.

From the previous analysis we understand that a key ingredient to obtain non-trivial scaling limits is the possibility to rearrange $K_J(x, y)$ as a sum of Christoffel–Darboux kernels. Let us consider a subset J of energy levels with a block structure (the union of B blocks)

$$J = \bigcup_{j=0}^{B-1} \left[a_j^2 M \mathinner{.\,.} (a_j + r_j)^2 M\right). \tag{52}$$

Hereafter, we assume that the a_j's and r_j's are such that J is a union of B *disjoint* blocks. The number of energy levels $N = |J|$ is

$$N = \sum_{j=0}^{B-1} (a_j + r_j)^2 - a_j^2. \tag{53}$$

Denote by Ψ_J the wave function representing N fermions with one fermion in each level $k \in J$. In formulae,

$$\Psi_J(x_1, \cdots, x_N) = \frac{1}{\sqrt{N!}} \det_{1 \leqslant i,j \leqslant N} \psi_{k_i}(x_j), \qquad \text{with } k_i \in J. \tag{54}$$

Then,

$$|\Psi_J(x_1, \cdots, x_N)|^2 = \frac{1}{N!} \det_{1 \leqslant i,j \leqslant N} K_J(x_i, x_j) \tag{55}$$

defines a determinantal point process on the line with kernel $K_J(x, y)$.

When M is large (a limit of large number of particles) the one-point function is

$$\varrho_1(x) \overset{M \to \infty}{\sim} \frac{1}{2\pi} \sum_{j=0}^{B-1} \left(\sqrt{(4(a_j + r_j)^2 M - x^2)_+} - \sqrt{(4a_j^2 M - x^2)_+} \right). \tag{56}$$

In the bulk, e.g. at $x = 0$,

$$\lim_{M \to \infty} \frac{1}{\sqrt{M}} \varrho_1(0) = \frac{R}{\pi}, \quad \text{with } R = \sum_{j=0}^{B-1} r_j. \tag{57}$$

It is not difficult to verify that the previous semiclassical considerations for the one-point function based on the correspondence principle (see equation (25)) carry over in the case of several blocks. For $B > 1$, when $M \to \infty$ the Wigner function in the phase space is uniform on B nested annuli; the projection onto the real line of the uniform density on nested annuli gives the number density (56). See figure 7.

The scaling limit of the kernel in the bulk is

$$\lim_{M \to \infty} \frac{1}{\varrho_1(0)} K_J \left(\frac{x}{\varrho_1(0)}, \frac{y}{\varrho_1(0)} \right) = k(x - y), \tag{58}$$

with

$$k(x - y) = \sum_{j=0}^{B-1} \frac{\sin \left(\frac{\pi r_j(x-y)}{2R} \right)}{\frac{\pi r_j(x-y)}{2R}} \frac{r_j}{R} \cos \frac{\pi(2a_j + r_j)(x - y)}{2R}. \tag{59}$$

There are two special block structures that give rise to α-determinantal processes, $\alpha = -\frac{1}{m}$ with m even or odd.

4.1. J of even type and $-\frac{1}{2B}$-determinantal processes

Suppose that $0 < a_0 < a_1 < \cdots < a_{B-1}$, and choose $r_0 = r_1 = \cdots = r_{B-1} = r$, so that $R = rB$. See top panel of figure 4. In formulae.

$$J = \left[a_0^2 M \, . \, . \, (a_0 + r)^2 M \right) \cup \left[a_1^2 M \, . \, . \, (a_1 + r)^2 M \right) \cup \cdots \cup \left[a_{B-1}^2 M \, . \, . \, (a_{B-1} + r)^2 M \right). \tag{60}$$

We say, for shortness, that J is of *even* type. Then,

$$k(x - y) = \frac{\sin \left(\frac{\pi(x-y)}{2B} \right)}{\frac{\pi(x-y)}{2B}} \cdot \frac{1}{B} \sum_{j=0}^{B-1} \cos \omega_j(x - y), \quad \text{where } \omega_j = \frac{\pi(2a_j + r)}{2rB}. \tag{61}$$

If $a_0, \ldots, a_{B-1}$ are sent (independently) to infinity, then for any $\sigma \in S_n$,

$$\lim_{a_0, \ldots, a_{B-1} \to \infty} \prod_{i=1}^{n} \frac{1}{B} \sum_{j=0}^{B-1} \cos \omega_j(x_{\sigma(i)} - x_i) = \left(\frac{1}{2B} \right)^{n-m(\sigma)}. \tag{62}$$
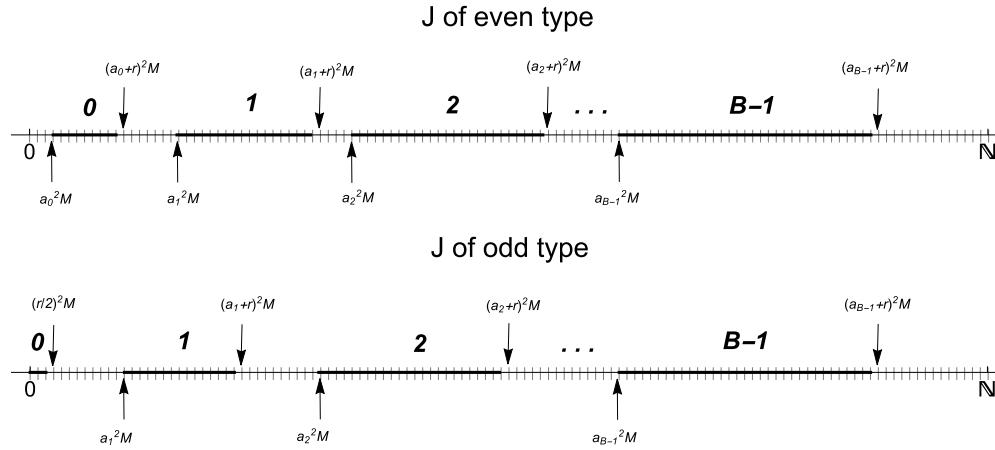
To see this, we expand the sum to get

J of even type

J of odd type

Figure 4. Scheme of the even type (top) and odd type (bottom) of subsets J's .

$$\prod_{i=1}^{n} \frac{1}{B} \sum_{j=0}^{B-1} \cos \omega_j (x_{\sigma(i)} - x_i) = \frac{1}{B^n} \sum_{j_1,\dots,j_n=0}^{B-1} \cos \omega_{j_1}(x_{\sigma(1)} - x_1) \cdots \cos \omega_{j_n}(x_{\sigma(n)} - x_n)$$

$$= \frac{1}{B^n} B^{m(\sigma)} \left(\frac{1}{2}\right)^{m_1-1} \cdots \left(\frac{1}{2}\right)^{m_{m(\sigma)}-1} + o(1), \tag{63}$$

where the factor $B^{m(\sigma)}$ is the number of ways of assigning frequencies ω_j, $j = 0, \dots, B-1$ to the $m(\sigma)$ cycles of σ. We conclude that, if J is of even type, then

$$\lim_{a_0,\dots,a_{B-1}\to\infty} \lim_{M\to\infty} \det_{1\leqslant i,j\leqslant n} \frac{1}{\varrho_1(0)^n} \det_{1\leqslant i,j\leqslant n} K_J\left(\frac{x_i}{\varrho_1(0)}, \frac{x_j}{\varrho_1(0)}\right) = \det_{\substack{-\frac{1}{2B} \\ 1\leqslant i,j\leqslant n}} \frac{\sin\frac{\pi}{2B}(x_i - x_j)}{\frac{\pi}{2B}(x_i - x_j)}. \tag{64}$$

4.2. J of odd type and $-\frac{1}{2B-1}$-determinantal processes

Suppose now that $0 = a_0 < a_1 < \cdots < a_{B-1}$, and choose $2r_0 = r_1 = \cdots = r_{B-1} = r$, i.e.

$$J = \left[0 \ldots (r/2)^2 M\right) \cup \left[a_1^2 M \ldots (a_1 + r)^2 M\right) \cup \cdots \cup \left[a_{B-1}^2 M \ldots (a_{B-1} + r)^2 M\right) \tag{65}$$

so that $R = r(B - 1/2)$. We say that J is of *odd* type. See bottom panel of figure 4. Then,

$$k(x - y) = \frac{\sin\left(\frac{\pi(x-y)}{2B-1}\right)}{\frac{\pi(x-y)}{2B-1}} \cdot \frac{1}{B - \frac{1}{2}} \left(\frac{1}{2} + \sum_{j=1}^{B-1} \cos\omega_j(x-y)\right), \quad \text{where } \omega_j = \frac{\pi(2a_j + r)}{2r(B - \frac{1}{2})}. \tag{66}$$

One can check that, in the limit of large $a_1, \dots, a_{B-1}$, for any $\sigma \in S_n$,

$$\lim_{a_1,\dots,a_{B-1}\to\infty} \prod_{i=1}^{n} \frac{1}{B - \frac{1}{2}} \left(\frac{1}{2} + \sum_{j=1}^{B-1} \cos\left(\omega_j(x_{\sigma(i)} - x_i)\right)\right) = \left(\frac{1}{2B-1}\right)^{n-m(\sigma)}. \tag{67}$$

The conclusion is that, if J has B blocks and is of odd type, then

$$\lim_{a_1\ldots,a_{B-1}\to\infty} \lim_{M\to\infty} \det_{1\leqslant i,j\leqslant n} \frac{1}{\varrho_1(0)^n} \det_{1\leqslant i,j\leqslant n} K_J\left(\frac{x_i}{\varrho_1(0)}, \frac{x_j}{\varrho_1(0)}\right) = \det_{-\frac{1}{2B-1}} \frac{\sin\frac{\pi}{2B-1}(x_i - x_j)}{\frac{\pi}{2B-1}(x_i - x_j)}. \tag{68}$$

5. Summary and remarks

We can summarise the findings of the previous sections as follows.

Theorem. *Let $K_J(x, y)$ be a kernel where J has B blocks as above. Consider the block projection process with correlation kernel $\varrho_1(0)^{-1}K_J\left(\varrho_1(0)^{-1}x, \varrho_1(0)^{-1}y\right)$. Then, in the limit $M \to \infty$ (first) and $a_i \to \infty$, the process converges to the α-determinantal process with kernel $\frac{\sin\pi\alpha(x-y)}{\pi\alpha(x-y)}$ (the union of $-1/\alpha$ rescaled sine processes). The parameter $\alpha < 0$ is $\alpha = -\frac{1}{2B}$ or $\alpha = -\frac{1}{2B-1}$ depending on whether J is of even or odd type, respectively. The convergence is in the sense of weak convergence of point processes.*

Note that the correlation functions $\varrho_n(x_1, \ldots, x_n) = \det_{\alpha} \frac{\sin\pi\alpha(x-y)}{\pi\alpha(x-y)}$ are bounded

$$\varrho_n(x_1, \ldots, x_n) \leqslant 1, \tag{69}$$

and hence determine uniquely the point process [26]. This limit process is translation invariant, and standard quantities of interest in the theory of point processes can be investigated.

5.1. Pair statistics and number variance

The pair statistics in Fourier space is traditionally studied by looking at properties of the *structure factor* defined (for a process with unit density) as [40, 44]

$$S(k) = 1 + \hat{h}(k), \tag{70}$$

where $\hat{h}(k)$ is the Fourier transform of the *total* or *connected correlation function*

$$h(r) = \varrho_2(x_1, x_2) - 1, \qquad r = x_1 - x_2. \tag{71}$$

For the process with correlation functions $\det_{\alpha} \frac{\sin\pi\alpha(x-y)}{\pi\alpha(x-y)}$ it is easy to calculate

$$S(k) = \begin{cases} \frac{|k|}{2\pi|\alpha|} & \text{if } |k| \leqslant 2\pi|\alpha| \\ 1 & \text{if } |k| > 2\pi|\alpha| \end{cases}. \tag{72}$$

We remark that, as the number of blocks B increases, we obtain a Poisson process, as expected (superposition of a large number of independent spectra [3]). Indeed, when $B \to \infty$, $\alpha \to 0$ and $\frac{\sin\pi\alpha(x-y)}{\pi\alpha(x-y)} \to 1$ for all x and y. Consequently, $\varrho_2(x, y) \to 1$ and hence $h(r) \to 0$, leading to $S(k) = 1$ (the structure factor of a Poisson process).

As already discussed, the convergence of the correlation functions when $a_i \to \infty$ is not pointwise. This is quite clear, as the cosine factors in $k(x - y)$ oscillates with high frequency. See figure 5. To illustrate better this point we consider the *number variance*, i.e. the variance of the number of fermions in a box $[-L/2, L/2]$ in the bulk when the quantum state of the fermions is $\Psi_J(x_1, \ldots, x_N)$. The expected number of particles is

$$E\left(\#\left[-\frac{L}{2}, \frac{L}{2}\right]\right) = \int_{-L/2}^{L/2} K_J(x, x)\mathrm{d}x. \tag{73}$$

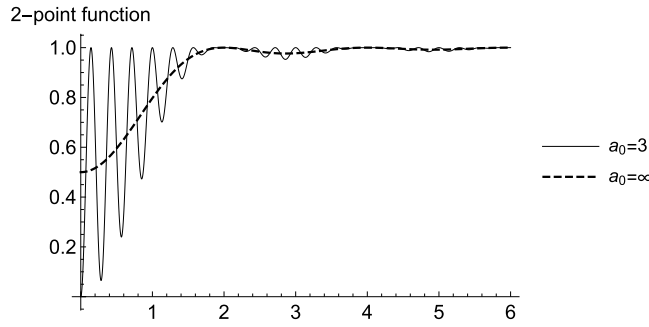**Figure 5.** Comparison of the two-point correlation function of a block projection process and its limit. Here $B = 1$, and the limit process is α-determinantal with $\alpha = -1/2$.

In the scaling limit in the bulk, the process becomes translation invariant and

$$\lim_{M \to \infty} E\left(\# \left[-\frac{L}{2\varrho_1(0)}, \frac{L}{2\varrho_1(0)} \right] \right) = L. \tag{74}$$

Standard manipulations give a formula for the variance in terms of the kernel $K_J(x, y)$:

$$\mathrm{Var}\left(\# \left[-\frac{L}{2}, \frac{L}{2} \right] \right) = \int_{-L/2}^{L/2} K_J(x, x)\mathrm{d}x - \iint_{-L/2}^{L/2} K_J(x, y)^2\mathrm{d}x\mathrm{d}y. \tag{75}$$

Taking the limit $M \to \infty$,

$$\lim_{M \to \infty} \mathrm{Var}\left(\# \left[-\frac{L}{2\varrho_1(0)}, \frac{L}{2\varrho_1(0)} \right] \right) = L - \iint_{-L/2}^{L/2} k(x - y)^2\mathrm{d}x\mathrm{d}y, \tag{76}$$

and, for large a_i's, the weak convergence of the process implies

$$\lim_{a_i \to \infty} \lim_{M \to \infty} \mathrm{Var}\left(\# \left[-\frac{L}{2\varrho_1(0)}, \frac{L}{2\varrho_1(0)} \right] \right) = L + \alpha \iint_{-L/2}^{L/2} \left(\frac{\sin \alpha\pi(x - y)}{\alpha\pi(x - y)} \right)^2 \mathrm{d}x\mathrm{d}y \tag{77}$$

where $\alpha = -\frac{1}{2B}$ or $\alpha = -\frac{1}{2B-1}$, if J is of even or odd type, respectively. For an illustration of this convergence, see figure 6.

In particular, in the limit $M \to \infty$ and $a_i \to \infty$, the number variance has the asymptotic expansions

$$\mathrm{Var}\left(\# \left[-\frac{L}{2\varrho_1(0)}, \frac{L}{2\varrho_1(0)} \right] \right) \sim \begin{cases} L + \alpha L^2 - \frac{1}{18}\pi^2\alpha^3 L^4 + \frac{2}{675}\pi^4\alpha^5 L^6 + \cdots & \text{as } L \to 0 \\ -\frac{1}{\alpha\pi^2}\left(\log L + \log(-2\pi\alpha) + 1 + \gamma_E + \cdots \right) & \text{as } L \to \infty \end{cases}, \tag{78}$$

where $\gamma_E = 0.577\,215\ldots$ is the Euler–Mascheroni constant. For $\alpha = -1$, the second line reduces to the well-known Dyson–Mehta result for the GUE [35].

5.2. Heuristic discussion and extension to other models

At this stage one may ask for a semiclassical explanation of the convergence of the fermion processes to α-determinantal processes. It is known that fermions generically display 'Friedel oscillations' [17–19] in the particle density and correlation functions with a wave
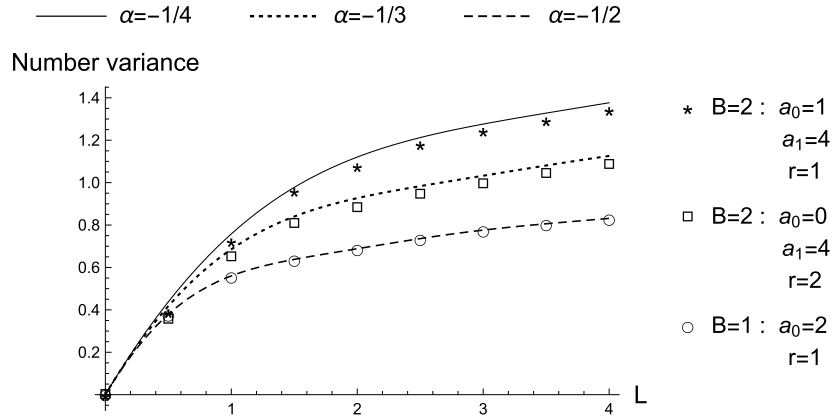
Figure 6. Number variance. The symbols comes from numerical integrations of (76) while the lines are the corresponding limits (77). Even for moderate values of the *a*'s the agreement is fairly good.

vector determined by a combination of Fermi surface effects and many-body effects. In the one-dimensional setting of non-interacting particles considered in this paper, the oscillations described by the sine kernel are simply a consequence of the sharpness of the Fermi surface (here points) at zero temperature. In the ground state, the momenta are in the Fermi sphere (interval) with edges $\pm\sqrt{N}$, and in the bulk the correlation kernel is the Fourier transform of the indicator function of that interval, hence the sine kernel with frequency $(1/2)\sqrt{N}$.

For $B > 1$ blocks, the Fermi sphere, i.e. the set of momenta $p \in \mathbb{R}$ for the wavefunction Ψ_J, is rather the union of Fermi shells (disjoint intervals). Oscillations occur in the correlation functions in the bulk, and their frequencies is related to the size of the Fermi shells. More precisely, if J is of even type, the set of possible momenta consists of $2B$ shells symmetric with respect to $p = 0$. If J is of odd type, there are $2B - 1$ intervals (one containing the origin) of possible values for the momenta. In both cases, when $M \to \infty$ each interval has the same length $\sim r\sqrt{M}$. This also explains why in the odd type we choose $r_0 = r/2$. In the scaling limit in the bulk, to each Fermi shell corresponds a correlation kernel with frequency given by $(1/2) \times r\sqrt{M}$; for large a_i's, the distance between the Fermi shells increases, and the oscillations of the kernels are asymptotically independent so that the process in the bulk becomes a superposition of independent sine processes with the same frequency. A glance at figure 7 may be helpful.

In fact, the reader may have recognised in the computation of the kernel $k(x - y)$ steps similar to the calculation of diffraction/interference patterns in wave optics [42]. For a single slit of width r (ground state $a = 0$) the far-field intensity distribution is proportional to $\left(\frac{\sin \pi rz}{\pi rz}\right)^2$. For two slits of width r at distance a (J of even type with one block) the interference pattern shows periodic fringes superimposed to the diffraction pattern $\left(\frac{\sin \pi rz}{\pi rz}\right)^2 \cos^2(\pi a)$ However, if the slits are too far apart (i.e. when $a \to \infty$), the waves coming from the two slits do not interfere, no fringes will be seen and the intensity distribution will be just the incoherent sum of the diffraction patterns from each individual slit. This easily extends to a generic number of slits.

These semiclassical considerations are also relevant in other block projection processes with correlation kernel
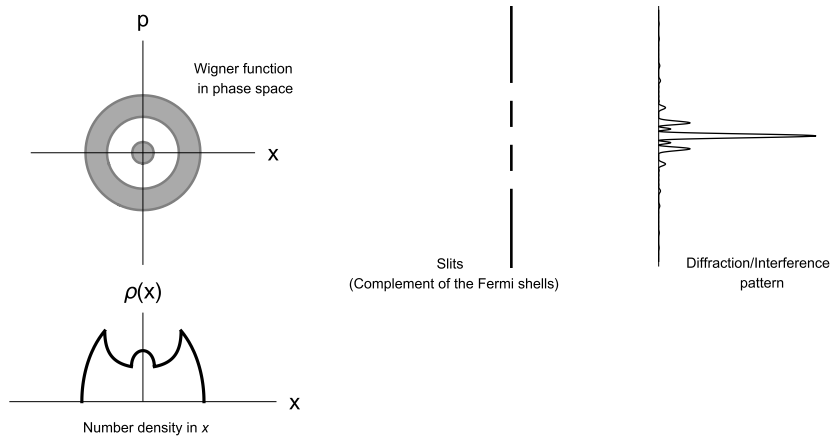
Figure 7. Scheme of the semiclassical considerations based on the correspondence principle. For large *M*, the Wigner function is uniform on a set of nested annuli. The projection onto the *x*-axis is the limit number density. The intersections of the annuli with the *p*-axis define the Fermi shells. The Fourier transform of the Fermi shells gives a sum of sine processes that become independent when the shells are far apart. A close analogy can be drawn with the diffraction patterns in wave optics.

$$K_J(x,y) = \sum_{k \in J} \overline{\psi_k}(x)\psi_k(y),$$

(79)

where ψ_k form an orthonormal basis of some L^2 space. For instance, one can consider the free fermions on the circle, i.e. block projection processes constructed using the family of trigonometric polynomials $\psi_k(x) = e^{ikx}$, $k \in \mathbb{Z}$. (These processes appeared under the name of 'Fermi shell models' in a work by Torquato, Scardicchio and Zachary [44].) It is easy to see that the scaling limit in the bulk, in the limit of blocks very far apart the is a superposition of independent sine processes.

As an alternative heuristics, one can imagine the Hermite block projection process as a complex Hermite block projection process conditioned to be real [25]. This gives a nice heuristic explanation as to why we see a superposition of independent sine processes in the limit when we are inside the inner radius of the annulus, coming from above and below (and becoming independent when $a \to \infty$). Similarly, in the circular case one should consider the block Ginibre process constructed using monomials z^k, $k \in \mathbb{Z}$; then, constrained to the unit circle this is the block trigonometric process, and we see asymptotic superposition of sine processes as expected.

5.3. Another α-determinantal process from random matrices of finite size

The α-determinantal processes described in this paper arise as scaling limits of block projection processes. In particular, the limit processes describe configurations of an infinite number of particles (superposition of sine kernels). It is natural to ask whether it is possible to get (in a non-trivial way) α-determinantal processes out of eigenvalues of random matrices of *finite* size. In fact, one example of such a construction can be read off from an intriguing decoupling phenomenon for power of random unitary matrices discovered by Rains [38, 39]. Let *m* and *N* be a positive integers with $m \leqslant N$, and let *U* be a random unitary matrix from the Haar

measure on $\mathcal{U}(mN)$. Then, the eigenvalues of U^m are exactly distributed as the union of eigenvalues of m independent unitary matrices $U_1, \ldots, U_m$ chosen in $\mathcal{U}(N)$.

It is a classical fact that the eigenvalues of random unitary matrices form a determinantal process on the unit circle. Set $S_n(z) = \frac{1}{2\pi} \frac{\sin(nz/2)}{\sin(z/2)}$, and denote by $x_1, \ldots, x_{mN}$ the eigenphases of a random unitary U of size mN. Then, the law of $x_1, \ldots, x_{mN}$ defines a determinantal process with kernel $S_{mN}(x - y)$. Rains' theorem can be restated by saying that the point configuration of mth powers $x_1^m, \ldots, x_{mN}^m$ is the union of m independent determinantal processes with kernel $S_N(x - y)$. Alternatively—and this is perhaps not so well-known—the mth powers $x_1^m, \ldots, x_{mN}^m$ form an α-determinantal process with $\alpha = -\frac{1}{m}$ and kernel $S_{mN}(x - y)$.

Similar results hold for the eigenvalue processes of the other classical compact groups [39], and have been recently extended to a class of rotation invariant determinantal processes in the complex plane by Dubach [10]. It remains an open problem to generalise this construction to other matrix ensembles without rotation symmetry.

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