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Exit problems associated with finite reflection groups

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Abstract. We obtain a formula for the distribution of the first exit time of Brownian motion from a fundamental region associated with a finite reflection group. In the type A case it is closely related to a formula of de Bruijn and the exit probability is expressed as a Pfaffian. Our formula yields a generalisation of de Bruijn's. We derive large and small time asymptotics, and formulas for expected first exit times. The results extend to other Markov processes. By considering discrete random walks in the type A case we recover known formulas for the number of standard Young tableaux with bounded height.

1. Introduction

The reflection principle is a protean concept which has given rise to many investigations in probability and combinatorics. Its most famous embodiment may be the ballot problem of counting the number of walks with unit steps staying above the origin. In the context of a one-dimensional Brownian motion $(B_t, t \geq 0)$ with transition density $p_t(x, y)$, the reflection principle gives a simple expression for the transition density $p_t^*(x, y)$ of the Brownian motion started in $(0, \infty)$ and killed when it first hits zero:

$$p_t^*(x, y) = p_t(x, y) - p_t(x, -y). \tag{1.1}$$

The exit probability is recovered by integrating over $y > 0$. If $\mathbb{P}_x$ denotes the law of B started at $x > 0$ and T is the first exit time from $(0, \infty)$, then

$$\mathbb{P}_x(T > t) = \mathbb{P}_x(B_t > 0) - \mathbb{P}_x(B_t < 0). \tag{1.2}$$

The formula (1.1) extends to the much more general setting of Brownian motion in a fundamental region associated with a finite reflection group. For example, if B is a Brownian motion in $\mathbb{R}^n$ with transition density $p_t(x, y)$ and $C = \{x \in \mathbb{R}^n : x_1 > x_2 > \cdots > x_n\}$ then the transition density of the Brownian motion, killed when it first exits C, is given by

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$$p_t^*(x,y) = \sum_{\pi\in\mathfrak{S}_n} \varepsilon(\pi) p_t(x,\pi y), \tag{1.3}$$

where $\pi y = (y_{\pi(1)}, \dots, y_{\pi(n)})$ and $\varepsilon(\pi)$ denotes the sign of π. Equivalently,

$$p_t^*(x,y) = (2\pi t)^{-n/2} \det[\exp(-(x_i - y_j)^2/2t)]_{i,j=1}^n. \tag{1.4}$$

This is referred to as the 'type A' case and the associated reflection group is isomorphic to $\mathfrak{S}_n$. The formula (1.4) is a special case of a more general formula due to Karlin and McGregor [17]; it can be verified in this setting by noting that the right-hand-side satisfies the heat equation with appropriate boundary conditions. Integrating (1.3) over $y \in C$ yields a formula for the exit probability

$$\mathbb{P}_x(T > t) = \sum_{\pi\in\mathfrak{S}_n} \varepsilon(\pi)\mathbb{P}_x(B_t \in \pi C). \tag{1.5}$$

This formula involves some complicated multi-dimensional integrals, but thanks to an integration formula of de Bruijn [6], it can be re-expressed as a Pfaffian which only involves one-dimensional integrals. More precisely, if we set $\gamma(a) = \sqrt{\frac{2}{\pi}} \int_0^a e^{-y^2/2} dy$ and $p_{ij} = \gamma\left(\frac{x_i - x_j}{\sqrt{2t}}\right)$ then, for $x \in C$,

$$\mathbb{P}_x(T > t) = \begin{cases} \mathrm{Pf}\left(p_{ij}\right)_{i,j\in[n]} & \text{if } n \text{ is even,} \\ \sum_{l=1}^n (-1)^{l+1}\, \mathrm{Pf}\left(p_{ij}\right)_{i,j\in[n]\setminus\{l\}} & \text{if } n \text{ is odd.} \end{cases} \tag{1.6}$$

(Observe that $p_{ji} = -p_{ij}$ since γ is an odd function and see appendix for a definition of the Pfaffian.) For example, when $n = 3$, we recover the simple formula

$$\mathbb{P}_x(T > t) = p_{12} + p_{23} - p_{13}, \tag{1.7}$$

which was obtained in [20] by direct reflection arguments.

The formula (1.3) extends naturally to Brownian motion in a fundamental region C associated with any finite reflection group (for discrete versions see Gessel and Zeilberger [12] and Biane [3]). As above, this can be integrated to give a formula for the exit probability involving multi-dimensional integrals. The main point of this paper is that there is an analogue of the simplified formula (1.6) in the general case which can be obtained directly. This leads to a generalisation of de Bruijn's formula and can be used to obtain asymptotic results as well as formulae for expected exit times. Our approach is not restricted to Brownian motion. For example, if we consider discrete random walks in the type A case we recover results of Gordon [14] and Gessel [11] on the number of standard Young tableaux with bounded height.

The outline of the paper is as follows. In the next section we introduce the reflection group setting and state the main results. These results involve a condition which we refer to as 'consistency'. This is discussed in detail for the various types of reflection groups in section 3. In section 4 we apply our results to give formulae for the exit probability of Brownian motion from a fundamental domain and use these formulae to obtain small and large time asymptotic expansions and to compute expected exit times. In section 5, we present a generalisation of de Bruijn's formula and in section 6 we describe some related combinatorics. The final section is devoted to proofs.

2. The main result

2.1. The reflection group setting

For background on root systems and finite reflection groups see, for example, [16]. Let V be a real Euclidean space endowed with a positive symmetric bilinear form (λ, μ). Let Φ be a (reduced) root system in V with associated reflection group W. Let Δ be a simple system in Φ with corresponding positive system Π and fundamental chamber $C = \{\lambda \in V : \forall\alpha \in \Delta,\ (\alpha, \lambda) > 0\}$. Denote the reflections in W by s_α $(\alpha \in \Pi)$.

Definition 2.1. A subset of V is said to be **orthogonal** if its distinct elements are pairwise orthogonal. If $E \subset V$, we will denote by $\mathcal{O}(E)$ the set of all orthogonal subsets of E.

Definition 2.2 (Consistency).

- We will say that $I \subset \Pi$ satisfies hypothesis (C1) if there exists $J \in \mathcal{O}(\Delta \cap I)$ such that if $w \in W$ with $J \subset wI \subset \Pi$ then $wI = I$.
- We will say that $I \subset \Pi$ satisfies hypothesis (C2) if the restriction of the determinant to the subgroup $U = \{w \in W : wI = I\}$ is trivial, ie $\forall w \in U$, $\varepsilon(w) = \det w = 1$.
- I will be called **consistent** if it satisfies (C1) and (C2).

Suppose $I \subset \Pi$ is consistent. Set $W^I = \{w \in W : wI \subset \Pi\}$ and $\mathcal{I} = \{wI : w \in W^I\}$. The hypothesis (C2) makes it possible to attribute a sign to every element of $\mathcal{I}$ by setting $\varepsilon_A := \varepsilon(w)$ for $A \in \mathcal{I}$, where w is any element of W^I with $wI = A$.

For example, $I = \Delta$ is consistent with $W^I = U = \{\mathrm{id}\}$ and $\mathcal{I} = \{\Delta\}$.

Section 3 will be devoted to a study of the consistency condition in the different types of root systems. Most root systems will turn out to possess a non-trivial (and useful) consistent subset $I \subset \Pi$.

2.2. The exit problem

Let $I \subset \Pi$ be consistent, and define ε_A for $A \in \mathcal{I}$ as above. Let $X = (X_t, t \geq 0)$ be a standard Brownian motion in V and write $\mathbb{P}_x$ for the law of X started at $x \in C$. For $\alpha \in \Pi$, set $T_\alpha = \inf\{t \geq 0 : (\alpha, X_t) = 0\}$. For $A \subset \Pi$ write $T_A = \min_{\alpha\in A} T_\alpha$, and set $T = T_\Delta = \inf\{t \geq 0 : X_t \notin C\}$.

Denote by $p_t(x, y)$ (respectively $p_t^*(x, y)$) the transition density of X (respectively that of X started in C and killed at time T). The analogue of the formula (1.3) in this setting is

$$p_t^*(x, y) = \sum_{w\in W} \varepsilon(w) p_t(x, wy), \tag{2.1}$$

which can be integrated to obtain

$$\mathbb{P}_x(T > t) = \sum_{w\in W} \varepsilon(w)\mathbb{P}_x(X_t \in wC). \tag{2.2}$$

A discrete version of this formula was obtained by Gessel and Zeilberger [12] and Biane [3]; it is readily verified in the continuous setting by observing that the expression given satisfies the heat equation with appropriate boundary conditions. As remarked in the introduction, this formula typically involves complicated multi-dimensional integrals. Our main result is the following alternative.

Proposition 2.3.

$$\mathbb{P}_x(T > t) = \sum_{A \in \mathcal{I}} \varepsilon_A \, \mathbb{P}_x(T_A > t). \tag{2.3}$$

In fact, we will prove Proposition 2.3 in the following, slightly more general, context. Let $X = (X_t, t \geq 0)$ be a Markov process with W-invariant state space $E \subset V$, infinitesimal generator $\mathcal{L}$, and write $\mathbb{P}_x$ for the law of the process started at x. Assume that the law of X is W-invariant, that is,

$$\mathbb{P}_x \circ (wX)^{-1} = \mathbb{P}_{wx} \circ X^{-1},$$

and that X is sufficiently regular so that:

(i) $u_I(x,t) = \mathbb{P}_x(T_I > t)$ satisfies the boundary-value problem:

$$\frac{\partial u_I}{\partial t} = \mathcal{L} u_I \qquad \begin{cases} u_I(x,0) = 1 & x \in O = \{\lambda \in V : \forall \alpha \in I, \ (\alpha, \lambda) > 0\}, \\ u_I(x,t) = 0 & x \in \partial O. \end{cases} \tag{2.4}$$

(ii) $u(x,t) = \mathbb{P}_x(T > t)$ is the *unique* solution to

$$\frac{\partial u}{\partial t} = \mathcal{L} u \qquad \begin{cases} u(x,0) = 1 & x \in C, \\ u(x,t) = 0 & x \in \partial C. \end{cases} \tag{2.5}$$

These hypotheses are satisfied if X is a standard Brownian motion in V or, in the crystallographic case, a continuous-time W-invariant simple random walk.

Note that for $I = \Delta$, the sum in (2.3) has only one term and the formula is a tautology. However, as we shall see, in general we can find more interesting and useful choices of I.

Remark 2.1. *There are explicit formulae of a different nature for the distribution of the exit time from a general convex cone $C \subset \mathbb{R}^k$. Typically, these are expressed as infinite series whose terms involve eigenfunctions of the Laplace-Beltrami operator on $C \cap S^{k-1}$ with Dirichlet boundary conditions. See, for example, [4], [2] and references therein.*

2.3. The orthogonal case

If I is orthogonal, the summation in (2.3) is over orthogonal subsets of Π, and Proposition 2.3 is therefore most effective when X has independent components in orthogonal directions. In this case, (2.3) becomes:

$$\mathbb{P}_x(T > t) = \sum_{A \in \mathcal{I}} \varepsilon_A \prod_{\alpha \in A} \mathbb{P}_x(T_\alpha > t), \tag{2.6}$$

where $\mathbb{P}_x(T_\alpha > t) = \mathbb{P}_x((X_t, \alpha) > 0) - \mathbb{P}_x((X_t, \alpha) < 0)$. For example, if X is Brownian motion, we have

$$\mathbb{P}_x(T > t) = \sum_{A \in \mathcal{I}} \varepsilon_A \prod_{\alpha \in A} \gamma\left(\hat{\alpha}(x)/\sqrt{t}\right), \tag{2.7}$$

where $\hat{\alpha}(x) = (\alpha, x)/|\alpha|$ and $\gamma(a) = \sqrt{\frac{2}{\pi}} \int_0^a e^{-y^2/2} dy$.

Consider the polynomial $Q \in \mathbb{Z}[X_\alpha,\ \alpha \in \Pi]$ defined by

$$Q = \sum_{A \in \mathcal{I}} \varepsilon_A \prod_{\alpha \in A} X_\alpha. \tag{2.8}$$

Then $\mathbb{P}_x(T > t)$ is equal to the polynomial Q evaluated at $\mathbb{P}_x(T_\alpha > t),\ \alpha \in \Pi$. Note that the polynomial Q is homogeneous of degree $|I|$. A useful property of Q is the following, which we record here for later reference.

Proposition 2.4. *For $x \in V$, set $P(x) = Q((\alpha, x),\ \alpha \in \Pi)$. If $I \neq \Pi$, then $P = 0$.*

2.4. A dual formula

In the orthogonal case, there is an analogue of the formula (2.6) for the complementary probability $\mathbb{P}_x(T \leq t)$. This will prove to be useful when analyzing the small time behaviour (see Section 4.6.2).

For $\alpha \in \Delta,\ B \in \mathcal{O}(\Pi)$, define $\alpha.B \in \mathcal{O}(\Pi)$ by:

$$\alpha.B = \begin{cases} B & \text{if } \alpha \in B; \\ \{\alpha\} \cup B & \text{if } \alpha \in B^\perp; \\ s_\alpha B & \text{otherwise.} \end{cases}$$

We can then define the "length" $l(B)$ for $B \in \mathcal{O}(\Pi)$ by:

$$l(B) = \inf\{l \in \mathbb{N} :\ \exists \alpha_1, \alpha_2, \ldots, \alpha_l \in \Delta,\ B = \alpha_l \ldots \alpha_2.\alpha_1.\emptyset\}. \tag{2.9}$$

Proposition 2.5. *For all $B \in \mathcal{O}(\Pi)$, $l(B) < \infty$. In other words, any $B \in \mathcal{O}(\Pi)$ can be obtained from the empty set by successive applications of the simple roots.*

Proposition 2.6. *Suppose I is consistent and orthogonal. Then,*

$$\mathbb{P}_x(T \leq t) = \sum_{B \in \mathcal{O}(\Pi) \setminus \{\emptyset\}} (-1)^{l(B)-1}\, \mathbb{P}_x[\forall \beta \in B,\ T_\beta \leq t]. \tag{2.10}$$

If we introduce the polynomial $R \in \mathbb{Z}[X_\alpha,\ \alpha \in \Pi]$,

$$R = \sum_{B \in \mathcal{O}(\Pi) \setminus \{\emptyset\}} (-1)^{l(B)-1} \prod_{\alpha \in B} X_\alpha, \tag{2.11}$$

then (2.10) is essentially equivalent to the following relation between Q and R:

$$1 - Q\left(1 - X_\alpha,\ \alpha \in \Pi\right) = R\left(X_\alpha,\ \alpha \in \Pi\right).$$

Note that R is not homogeneous.

2.5. The semi-orthogonal case

Definition 2.7. We say $E \subset V$ is **semi-orthogonal** if it can be partitioned into blocks (ρ_i) such that $\rho_i \perp \rho_j$ for $i \neq j$ and each ρ_i is either a singlet or a pair of vectors whose mutual angle is $3\pi/4$. The set of the blocks ρ_i will be denoted by E^*.

Remark 2.2. *A prototypical pair of vectors in a semi-orthogonal subset is $\{e_1 - e_2, e_2\}$, where (e_1, e_2) is orthonormal.*

If I is consistent and semi-orthogonal and if X has independent components in orthogonal directions, the formula (2.3) becomes:

$$\mathbb{P}_x(T > t) = \sum_{A \in \mathcal{I}} \varepsilon_A \prod_{\rho \in A^*} \mathbb{P}_x(T_\rho > t). \tag{2.12}$$

Call Π' the set of pairs of positive roots whose mutual angle is $3\pi/4$. The relevant polynomial to consider is $S \in \mathbb{Z}[X_\alpha,\ \alpha \in \Pi;\ X_{\{\alpha,\beta\}},\ \{\alpha, \beta\} \in \Pi']$,

$$S = \sum_{A \in \mathcal{I}} \varepsilon_A \prod_{\rho \in A^*} X_\rho. \tag{2.13}$$

Proposition 2.8. *Suppose $2|I| < |\Pi|$. For $x \in V$, the evaluation of S with $X_\alpha = (\alpha, x),\ \alpha \in \Pi$ and $X_{\{\alpha,\beta\}} = (\alpha, x)(\beta, x)(s_\alpha \beta, x)(s_\beta \alpha, x),\ \{\alpha, \beta\} \in \Pi'$, is equal to zero.*

3. Consistency

Lemma 3.1. *Suppose there exists $J \in \mathcal{O}(\Delta)$ which is uniquely extendable to a maximal orthogonal (resp. semi-orthogonal) subset $I \subset \Pi$, maximal meaning that there is no orthogonal (resp. semi-orthogonal) subset strictly larger than I. In this case, I satisfies condition* (C1).

Proof. If $J \subset wI \subset \Pi$ then wI is a maximal orthogonal (resp. semi-orthogonal) subset of Π and the unique extension property says that $wI = I$. $\square$

3.1. The dihedral groups

The dihedral group $I_2(m)$ is the group of symmetries of a regular m-sided polygon centered at the origin. It is a reflection group acting on $V = \mathbb{R}^2 \simeq \mathbb{C}$. Define $\beta = i$, $\alpha_l = e^{il\pi/m}(-\beta)$ for $1 \leq l \leq m$ and $\alpha = \alpha_1$. Then we can take $\Pi = \{\alpha_1, \ldots, \alpha_m\}$ and $\Delta = \{\alpha, \beta\}$.

Set $I = \{\alpha\}$ if m is odd and $I = \{\alpha, \alpha' = e^{i\pi/2}\alpha\}$ if $m \equiv 2 \bmod 4$. Then I is orthogonal and consistent. In the first case, $\mathcal{I} = \{\{\alpha_1\}, \ldots, \{\alpha_m\}\}$ with $\varepsilon_{\{\alpha_i\}} = (-1)^{i-1}$. In the second case, $\mathcal{I} = \left\{\{\alpha_1, \alpha_1'\}, \ldots, \{\alpha_m, \alpha_m'\}\right\}$ and $\varepsilon_{\{\alpha_i, \alpha_i'\}} = (-1)^{i-1}$. With notations $X_j = X_{\alpha_j}$ and $X_j' = X_{\alpha_j'}$, the polynomial Q can be written

$$Q = \begin{cases} \sum_{j=1}^m (-1)^{j-1} X_j & \text{if } m \text{ is odd} \\ \sum_{j=1}^m (-1)^{j-1} X_j X_j' & \text{if } m \equiv 2 \mod 4\,. \end{cases} \tag{3.1}$$

3.2. The A_{k-1} case

Consider $W = \mathfrak{S}_k$ acting on $\mathbb{R}^k$ by permutation of the canonical basis vectors. Then we can take $V = \mathbb{R}^k$ or $\{x \in \mathbb{R}^k : x_1+\cdots+x_k = 0\}$, $\Pi = \{e_i - e_j,\ 1 \le i < j \le k\}$ and $\Delta = \{e_i - e_{i+1},\ 1 \le i \le k-1\}$.

The choice of I depends on the parity of k. If k is even, we take

$$I = \{e_1 - e_2, e_3 - e_4, \dots, e_{k-1} - e_k\}.$$

If k is odd, then

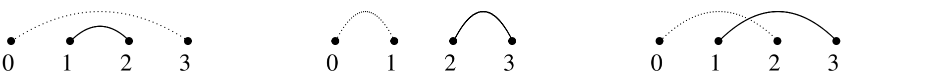
$$I = \{e_1 - e_2, e_3 - e_4, \dots, e_{k-2} - e_{k-1}\}.$$

Proposition 3.1. (i) *I is consistent and orthogonal.*

(ii) *The set $\mathcal{I}$ can be identified with the set $P_2(k)$ of partitions of $[k]$ into $k/2$ pairs if k is even and into $(k-1)/2$ pairs and a singlet if k is odd.*

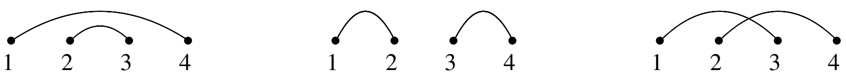
(iii) *Under this identification, the sign ε is just the parity of the number of crossings (if k is odd, we consider an extra pair made of the singlet and a formal dot 0 strictly at the left of 1 and use this pair to compute the number of crossings).*

The proof of this proposition will be provided in Section 7.1. In Figures 1 and 2, we give examples of the identification between $A \in \mathcal{I}$ and $\pi \in P_2(k)$, using the notation $c(\pi)$ for the number of crossings.



$\pi = \{\{1,2\},\{3\}\}$	$\pi = \{\{1\},\{2,3\}\}$	$\pi = \{\{1,3\},\{2\}\}$
$A = \{e_1 - e_2\}$	$A = \{e_2 - e_3\}$	$A = \{e_1 - e_3\}$
$c(\pi) = 0$	$c(\pi) = 0$	$c(\pi) = 1$

Fig. 1. Pair partitions and their signs for A_2



$\pi = \{\{1,4\},\{2,3\}\}$	$\pi = \{\{1,2\},\{3,4\}\}$	$\pi = \{\{1,3\},\{2,4\}\}$
$A = \{e_1 - e_4, e_2 - e_3\}$	$A = \{e_1 - e_2, e_3 - e_4\}$	$A = \{e_1 - e_3, e_2 - e_4\}$
$c(\pi) = 0$	$c(\pi) = 0$	$c(\pi) = 1$

Fig. 2. Pair partitions and their signs for A_3

Now, recall the polynomial Q defined in (2.8) and write for simplicity $X_{ij} = X_{e_i - e_j}$, $i < j$. Then,

$$Q = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\} \in \pi} X_{ij}, \tag{3.2}$$

which can be expressed as a Pfaffian (see Appendix),

$$Q = \begin{cases} \operatorname{Pf}\left(X_{ij}\right)_{i,j \in [k]} & \text{if } k \text{ is even ,} \\ \sum_{l=1}^{k} (-1)^{l+1} \operatorname{Pf}\left(X_{ij}\right)_{i,j \in [k] \setminus \{l\}} & \text{if } k \text{ is odd,} \end{cases} \tag{3.3}$$

with the convention that $X_{ij} = -X_{ji}$ for $i > j$.

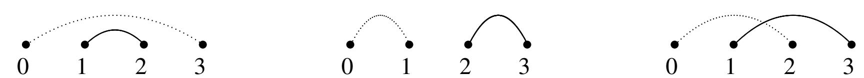
Remark 3.1. *It is interesting to make the combinatorial meaning of Proposition 3.1 explicit. Suppose k is even for simplicity. If* $\pi = \{\{j_1, j_1'\}, \dots, \{j_p, j_p'\}\}$ *is a pair partition of* $[k]$ *with* $j_i < j_i'$ *then we can define* $\sigma \in \mathfrak{S}_k$ *by* $\sigma(2i-1) = j_i, \sigma(2i) = j_i'$*. This definition depends on the numbering of the blocks of* π*, giving rise to* $(k/2)!$ *such permutations* σ*. The result is that they all have the same sign which is precisely* $(-1)^{c(\pi)}$*. If we order the blocks in such a way that* $j_1 < j_2 < \cdots < j_p$*, then we can be even more precise. Let* $i(\sigma)$ *denote the number of inversions of* σ *and* $b(\pi)$ *the number of bridges of* π*, that is of pairs* $i < l$ *with* $j_i < j_l < j_l' < j_i'$*. Then,*

$$i(\sigma) = c(\pi) + 2b(\pi). \tag{3.4}$$

3.3. The D_k case

We consider the group W of evenly signed permutations on $\{1, \dots, k\}$. More precisely, $f : \mathbb{R}^k \to \mathbb{R}^k$ is a sign flip with support $\overline{f}$ if $(fx)_i = -x_i$ when $i \in \overline{f}$ and $(fx)_i = x_i$ when $i \notin \overline{f}$. The elements of W are all $f\sigma$ where $\sigma \in \mathfrak{S}_k$ and f is a sign flip whose support has even cardinality. W is a reflection group and we take $V = \mathbb{R}^k$, $\Pi = \{e_i \pm e_j,\ 1 \le i < j \le k\}$ and $\Delta = \{e_1 - e_2, e_2 - e_3, \dots, e_{k-1} - e_k, e_{k-1} + e_k\}$.

For even k (resp. odd k), we take $I = \{e_1 \pm e_2, e_3 \pm e_4, \dots, e_{k-1} \pm e_k\}$ (resp. $I = \{e_2 \pm e_3, e_4 \pm e_5, \dots, e_{k-1} \pm e_k\}$). Proposition (3.1) is exactly the same in this case. The identification between $\mathcal{I}$ and $P_2(k)$ is performed as shown in the following examples:



$\pi = \{\{1,2\},\{3\}\}$ $\quad$ $\pi = \{\{1\},\{2,3\}\}$ $\quad$ $\pi = \{\{1,3\},\{2\}\}$

$A = \{e_1 - e_2, e_1 + e_2\}$ $\quad$ $A = \{e_2 - e_3, e_2 + e_3\}$ $\quad$ $A = \{e_1 - e_3, e_1 + e_3\}$

$c(\pi) = 0$ $\quad$ $c(\pi) = 0$ $\quad$ $c(\pi) = 1$

Fig. 3. Pair partitions and their signs for D_3

Writing $X_{ij} = X_{e_i - e_j} = -X_{ji},\ \bar{X}_{ij} = X_{e_i+e_j} = \bar{X}_{ji},\ i < j$, we have

$$Q = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\}\in\pi} X_{ij}\,\bar{X}_{ij}, \tag{3.5}$$

which can be expressed as a Pfaffian,

$$Q = \begin{cases} \mathrm{Pf}\left(X_{ij}\,\bar{X}_{ij}\right)_{i,j\in[k]} & \text{if } k \text{ is even },\\ \sum_{l=1}^{k}(-1)^{l+1}\,\mathrm{Pf}\,(X_{ij}\,\bar{X}_{ij})_{i,j\in[k]\setminus\{l\}} & \text{if } k \text{ is odd.}\end{cases} \tag{3.6}$$

3.4. The B_k case

W is the group of signed permutations on $\{1,\dots,k\}$, that is $W = \{f\sigma\ :\ \sigma \in \mathfrak{S}_k,\ \ f \text{ is a sign flip}\}$. W is a reflection group acting on $V = \mathbb{R}^k$ and the root system is determined by $\Pi = \{e_i - e_j,\ 1 \le i < j \le k;\ e_i,\ 1 \le i \le k\}$ and $\Delta = \{e_1 - e_2, e_2 - e_3, \dots, e_{k-1} - e_k, e_k\}$.

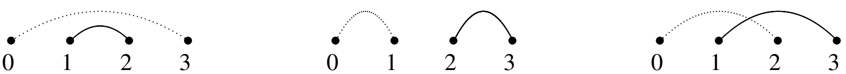
If k is even (resp. odd), set $I = \{e_1 - e_2, e_2, e_3 - e_4, e_4, \dots, e_{k-1} - e_k, e_k\}$ (resp. $I = \{e_1 - e_2, e_2, e_3 - e_4, e_4, \dots, e_{k-2} - e_{k-1}, e_{k-1}, e_k\}$). Then I is not orthogonal but only semi-orthogonal. Proposition (3.1) is still unchanged. The identification between $\mathcal{I}$ and $P_2(k)$ is performed as shown in Figure 4.

Writing $X_{ij} = X_{e_i - e_j} = -X_{ji},\ \hat{X}_{ij} = \hat{X}_{ji} = X_{e_j} = X_j,\ i < j$, we have

$$Q = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} X_{s(\pi)} \prod_{\{i<j\}\in\pi} X_{ij}\,\hat{X}_{ij}, \tag{3.7}$$

where $s(\pi)$ is the singlet of π, the term $X_{s(\pi)}$ being absent if k is even. (3.7) can be expressed as a Pfaffian,

$$Q = \begin{cases} \mathrm{Pf}\left(X_{ij}\,\hat{X}_{ij}\right)_{i,j\in[k]} & \text{if } k \text{ is even },\\ \sum_{l=1}^{k}(-1)^{l+1}\,X_l\,\mathrm{Pf}\,(X_{ij}\,\hat{X}_{ij})_{i,j\in[k]\setminus\{l\}} & \text{if } k \text{ is odd.}\end{cases} \tag{3.8}$$



$\pi = \{\{1,2\},\{3\}\}$ | $\pi = \{\{1\},\{2,3\}\}$ | $\pi = \{\{1,3\},\{2\}\}$

$A = \{e_1 - e_2, e_2, e_3\}$ | $A = \{e_1, e_2 - e_3, e_3\}$ | $A = \{e_1 - e_3, e_3, e_2\}$

$c(\pi) = 0$ | $c(\pi) = 0$ | $c(\pi) = 1$

Fig. 4. Pair partitions and their signs for B_3.

3.5. H_3 and H_4

The underlying Euclidean space is $\mathbb{R}^4$ identified with the quaternions $\mathbb{H}$ with standard basis $(1, i, j, k)$. We adopt the notations of [16] which we refer to for the definitions of the root systems in these cases. For H_3, we take $I = \{a - i/2 + bj, 1/2 + bi - aj, b + ai + j/2\}$ and for H_4, we take $I = \{-a + i/2 + bj, -1/2 - ai + bk, b + aj + k/2, bi - j/2 + ak\}$. Then I is orthogonal and consistent in both cases.

3.6. F_4

The underlying Euclidean space is $\mathbb{R}^4$ with standard basis (e_1, e_2, e_3, e_4) and again we refer to [16] for definition of the root system. We choose $I = \{e_2 - e_3, e_3, e_1 - e_4, e_4\}$. I is semi-orthogonal and consistent.

4. Applications to Brownian motion

4.1. Brownian motion in a wedge and the dihedral groups

In this case T is the exit time of a planar Brownian motion from a wedge of angle π/m:

$$C = \{re^{i\theta} : r \geq 0, 0 < \theta < \pi/m\} \subset \mathbb{C} \simeq \mathbb{R}^2.$$

Recall that $\alpha_l = e^{i\pi(l/m-1/2)}$ and $\alpha_l' = e^{i\pi/2}\alpha_l$.

Formula (2.3) reads

$$\mathbb{P}_x(T > t) = \begin{cases} \sum_{i=1}^m (-1)^{i-1} \mathbb{P}_x(T_{\alpha_i} > t) & \text{if } m \text{ is odd },\\ \sum_{i=1}^m (-1)^{i-1} \mathbb{P}_x(T_{\{\alpha_i, \alpha_i'\}} > t) & \text{if } m \equiv 2 \ (\bmod\ 4). \end{cases} \tag{4.1}$$

In the case of Brownian motion, the formulae (4.1) can be rewritten respectively as

$$\mathbb{P}_x(T > t) = \sum_{i=1}^m (-1)^{i-1} \gamma\left(\frac{-r\sin(\theta - i\pi/m)}{\sqrt{t}}\right), \tag{4.2}$$

and

$$\mathbb{P}_x(T > t) = \sum_{i=1}^m (-1)^{i-1} \gamma\left(\frac{-r\sin(\theta - i\pi/m)}{\sqrt{t}}\right) \gamma\left(\frac{r\cos(\theta - i\pi/m)}{\sqrt{t}}\right), \tag{4.3}$$

where $x = re^{i\theta} \in C$.

These should be compared with the formulae presented in [9]. If we take the previous formulae with $r = 1$, integrate over $\theta \in (0, \pi/m)$ and compute a Laplace transform, we recover results of [9] for m not a multiple of 4. However, the results in [9] are valid for all m. The case $m \equiv 0 \ (\bmod\ 4)$ will be discussed in Section 4.5.

General results on exit times from wedges can be found in the paper of Spitzer [21]. We recover Spitzer's results for the special angles discussed above.

4.2. The A_{k-1} case and non-colliding probability

The fundamental chamber is $C = \{x \in V : x_1 > x_2 > \cdots > x_k\}$ where $V = \mathbb{R}^k$ or $\{x \in \mathbb{R}^k : x_1 + \cdots + x_k = 0\}$. Thus T is the first 'collision time' between any two coordinates of X.

Using notation $\{i < j\} \in \pi$ to mean that $\{i, j\} \in \pi$ and $i < j$, formula (2.7) reads:

$$\mathbb{P}_x(T > t) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\} \in \pi} p_{ij}, \tag{4.4}$$

where $p_{ij} = \mathbb{P}_x(T_{e_i - e_j} > t) = \gamma\left(\frac{x_i - x_j}{\sqrt{2t}}\right)$.

When $k = 3$, the formula is

$$\mathbb{P}_x(T > t) = p_{12} + p_{23} - p_{13}. \tag{4.5}$$

When $k = 4$, the formula becomes

$$\mathbb{P}_x(T > t) = p_{12}p_{34} + p_{14}p_{23} - p_{13}p_{24}. \tag{4.6}$$

For odd k, we can isolate the singlet in (4.4) to deduce the following relation between the problem with k particles and the problem with $k - 1$ particles:

$$\mathbb{P}_x(T > t) = \sum_{l=1}^{k} (-1)^{l-1} \mathbb{P}_x(T_l > t), \tag{4.7}$$

where $T_l = \inf\{t : \exists\, i \neq j \in [k] \backslash \{l\},\ X_i(t) = X_j(t)\}$ so that $\mathbb{P}_x(\hat{T}_l > t)$ only depends on $(x_i)_{i \in [k] \backslash \{l\}}$.

Recalling definition and expression of the Pfaffian given in the Appendix, formula (4.4) reads:

$$\mathbb{P}_x(T > t) = \begin{cases} \operatorname{Pf}\left(p_{ij}\right)_{i,j \in [k]} & \text{if } k \text{ is even }, \\ \sum_{l=1}^{k} (-1)^{l+1} \operatorname{Pf}\left(p_{ij}\right)_{i,j \in [k] \backslash \{l\}} & \text{if } k \text{ is odd}, \end{cases} \tag{4.8}$$

with the convention that $p_{ji} = -p_{ij}$ for $i \leq j$. The merit of these formulae is to replace the alternating sums by closed-form expressions which are easier to compute (Pfaffians are just square roots of determinants).

4.3. The D_k case

The chamber is $C = \{x \in \mathbb{R}^k : x_1 > \cdots > x_{k-1} > |x_k|\}$. The formula (2.7) becomes:

$$\mathbb{P}_x(T > t) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\} \in \pi} \gamma\left(\frac{x_i - x_j}{\sqrt{2t}}\right) \gamma\left(\frac{x_i + x_j}{\sqrt{2t}}\right). \tag{4.9}$$

We can write it down in terms of Pfaffians:

$$\mathbb{P}_x(T > t) = \begin{cases} \operatorname{Pf}\left(p_{ij}\, \bar{p}_{ij}\right)_{i,j \in [k]} & \text{if } k \text{ is even }, \\ \sum_{l=1}^{k} (-1)^{l+1} \operatorname{Pf}\left(p_{ij}\, \bar{p}_{ij}\right)_{i,j \in [k] \backslash \{l\}} & \text{if } k \text{ is odd}, \end{cases} \tag{4.10}$$

where $\bar{p}_{ij} = \bar{p}_{ji} = \mathbb{P}_x(T_{e_i + e_j} > t) = \gamma\left(\frac{x_i + x_j}{\sqrt{2t}}\right)$.

4.4. The B_k case

The chamber is $C = \{x \in \mathbb{R}^k : x_1 > \cdots > x_{k-1} > x_k > 0\}$. The formula for B_k is:

$$\mathbb{P}_x(T > t) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \, \mathbb{P}_x(T_{e_{s(\pi)}} > t) \prod_{\{i<j\} \in \pi} \mathbb{P}_x(T_{e_i - e_j, e_j} > t),$$

where $s(\pi)$ is the singlet of π, the term $\mathbb{P}_x(T_{e_{s(\pi)}} > t)$ being absent when k is even.

The result is a polynomial involving the probabilities of exiting orthants associated to $\{e_i - e_j, e_j\}$. Those are wedges of angle $\pi/4$.

Those formulae can be rewritten in terms of Pfaffians:

$$\mathbb{P}_x(T > t) = \begin{cases} \mathrm{Pf}\,\left(\hat{p}_{ij}\right)_{i,j \in [k]} & \text{if } k \text{ is even }, \\ \sum_{l=1}^k (-1)^{l+1} \, p_l \, \mathrm{Pf}\,\left(\hat{p}_{ij}\right)_{i,j \in [k] \setminus \{l\}} & \text{if } k \text{ is odd,} \end{cases} \tag{4.11}$$

where $p_l = \mathbb{P}_x(T_{e_l} > t) = \gamma(x_l/\sqrt{t})$ and $\hat{p}_{ij} = -\hat{p}_{ji} = \mathbb{P}_x(T_{e_i - e_j, e_j} > t)$ for $i < j$.

4.5. Wedges of angle $\pi/4n$

First, let us pay attention to the case of an angle $\pi/4$ where $C = \{x_1 > x_2 > 0\}$. Consider the group $W = I_2(4)$ of isometries of the square: $W = \{\mathrm{id}, r, r^2, r^3, s_1, s_2, s_3, s_4\}$, r being the rotation of angle $\pi/2$ and s_1 (resp. s_2, s_3, s_4) the symmetry with respect to the line $y = -x$ (resp. $y = 0$, $y = x$, $x = 0$). By translation and W-invariance, formula (2.2) can be rewritten

$$\mathbb{P}_x(T > t) = \sum_{w \in W} \varepsilon(w) \mathbb{P}(X_t \in -w(x) + C). \tag{4.12}$$

Now, define regions E_i, $1 \le i \le 12$ as in Figure 5 and $P_i = \mathbb{P}(X_t \in E_i)$.

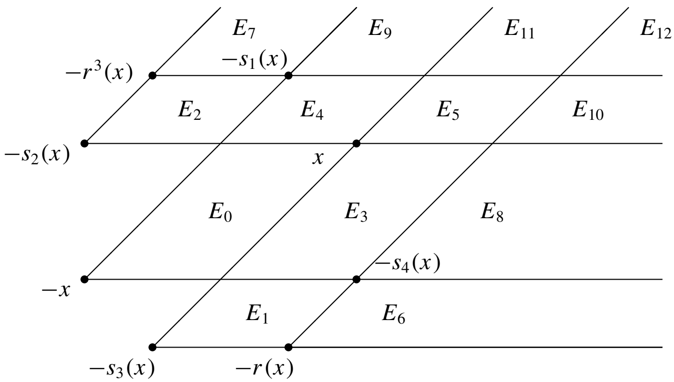


Fig. 5. The regions E_i

Then, we can decompose the terms in (4.12) into $\mathbb{P}(X_t \in -x+C) = P_0+P_3+P_4+P_5+P_8+P_9+P_{10}+P_{11}+P_{12}$, $\mathbb{P}(X_t \in -r(x)+C) = P_6+P_8+P_{10}+P_{12}$, $\mathbb{P}(X_t \in -r^2(x)+C) = P_5+P_{10}+P_{11}+P_{12}$, $\mathbb{P}(X_t \in -r^3(x)+C) = P_7+P_9+P_{11}+P_{12}$, $\mathbb{P}(X_t \in -s_1(x)+C) = P_9+P_{10}+P_{11}+P_{12}$, $\mathbb{P}(X_t \in -s_2(x)+C) = P_2+P_4+P_5+P_7+P_9+P_{10}+P_{11}+P_{12}$, $\mathbb{P}(X_t \in -s_3(x)+C) = P_1+P_3+P_5+P_6+P_8+P_{10}+P_{11}+P_{12}$, $\mathbb{P}(X_t \in -s_4(x)+C) = P_8+P_{10}+P_{12}$. In the end, terms cancel in an abundant way to result in

$$\mathbb{P}_x(T > t) = P_0 - P_1 - P_2 = P_0 - 2P_1, \tag{4.13}$$

since E_1 and E_2 are symmetric with respect to the origin. The interest of (4.13) is two-fold: first, the number of terms is significantly lower than in (4.12) and second, the integrals involved are over bounded regions. Indeed, write $x'_i = x_i/\sqrt{t}$ for simplicity, then

$$P_0 = \frac{1}{\pi}\int_0^{x'_1-x'_2} dy \int_{-x'_2}^{x'_2} dz\, e^{-((y+z)^2+z^2)/2},$$
$$P_1 = \frac{1}{2\pi}\int_{x'_1-x'_2}^{x'_1+x'_2} dy \int_{-x'_1}^{-x'_2} dz\, e^{-((y+z)^2+z^2)/2},$$

so that

$$\begin{aligned}\mathbb{P}_x(T > t) &= \frac{1}{\pi}\int_0^{x'_1-x'_2} dy \int_{-x'_2}^{x'_2} dz \;-\; \int_{x'_1-x'_2}^{x'_1+x'_2} dy \int_{-x'_1}^{-x'_2} dz\, e^{-((y+z)^2+y^2)/2} \\ &:= H(x'_1, x'_2). \end{aligned} \tag{4.14}$$

The integral formula defines H on $\mathbb{R}^2$ and implies that $H(y_1, -y_2) = H(y_1, y_2)$ and $H(y_2, y_1) = -H(y_1, y_2)$. Thus, expanding G in power series, we have

$$H(y_1, y_2) = \sum_{p,q\ge 0} a_{pq}\,(y_1\, y_2)^{2p+1}\,(y_1^{2q} - y_2^{2q}). \tag{4.15}$$

This type of result generalizes to angles $\pi/4n$ with the same kind of reasoning. Write $C_0 = \{re^{i\theta};\ r \ge 0,\ 0 \le \theta \le \pi/4n\}$ the fundamental chamber, $(C_i)_{1\le i\le 8n-1}$ the other chambers in counterclockwise order of appearance and x_i the W-image of $x \in C_0$ which lies in C_i. Call $\mathcal{P}(x, y)$ the parallelogram which has x and y as two opposite vertices and whose sides are parallel to the walls of the chamber C_0. Define $P_i = P_i(x,t) := \mathbb{P}_0(X_t \in \mathcal{P}(x_i, x_{4n-i}))$. Then, the formula is

$$\mathbb{P}_x(T > t) = P_0 + 2\sum_{i=1}^{2n-1} (-1)^i P_i.$$

4.6. Asymptotic expansions

We will now consider the asymptotic behaviour of $\mathbb{P}_x(T > t)$ in two different regimes where t is either large or small compared with the initial distance between x and the boundary ∂C.

4.6.1. Long time behaviour

We will suppose that $|\Pi|$ has the same parity as $|I|$, which is the case in all of our examples.

Proposition 4.1. *If I is consistent and orthogonal (or semi-orthogonal), the following expansion holds:*

$$\mathbb{P}_x(T > t) = h(x) \sum_{q \geq 0} E_q(x)\, t^{-(q+n/2)}, \tag{4.16}$$

where $n = |\Pi|$, $h(x) = \prod_{\alpha \in \Pi}(x, \alpha)$, $E_q(x)$ is a W-invariant polynomial of degree $2q$ and the series is convergent for all $x \in C$, $t > 0$. In particular, there exists a constant κ such that:

$$\mathbb{P}_x(T > t) \sim \frac{\kappa\, h(x)}{t^{n/2}} \text{ as } t \to \infty. \tag{4.17}$$

Remark 4.1. *The leading term in (4.17) was obtained in Grabiner [15] for the classical root systems of types A, B and D. More general results on expected exit times from cones and their moments can be found in [8], [7] and [4].*

Remark 4.2. *The polynomials E_q satisfy $\Delta_h E_{q+1} = -(q + n/2)\, E_q$, where*

$$\Delta_h f = \frac{1}{2}\Delta f + (\nabla \log h) \cdot \nabla f.$$

This follows from the fact that $u(x,t) = \mathbb{P}_x(T > t)$ satisfies the heat equation and $\Delta h = 0$.

For $I_2(m)$ with $m \neq 0 \mod 4$, one has $|\Pi| = m$ so that

$$\mathbb{P}_x(T > t) \sim \kappa\, \frac{h(x)}{t^{m/2}}.$$

Let us indicate how to compute the constant κ for odd $m = 2m' + 1$. We have $\kappa\, h(x) = a_{m'} \sum_1^m (-1)^{j-1}(x, \alpha_j)^m$. Suppose $x = e^{i\theta}$ then $(x, \alpha_j) = \sin(j\pi/m - \theta)$. Let $s = \pi/m - \theta$ go to 0 so that $h(x) \sim a\, s$ with $a = \prod_{j=1}^{m-1} \lambda_j = \prod_{j=1}^{m'} \lambda_j^2$, $\lambda_j = \sin(j\pi/m)$. The numbers $0, \lambda_j, -\lambda_j$, $1 \leq j \leq m'$ are the roots of the Tchebycheff polynomial T_m such that $T_m(\cos\theta) = \cos m\theta$. The leading coefficient of T_m being 2^{m-1}, we have

$$a = (-1)^{m'} 2^{1-m} T_m'(0) = m 2^{1-m}.$$

On the other hand, $\sum_1^m (-1)^{j-1}(x, \alpha_j)^m \sim b\, s$ with

$$b = m \sum_1^{m-1} (-1)^j \sin^{m-1}(j\pi/m) \cos(j\pi/m).$$

Thus, $\kappa = a_{m'} b/a$.

For A_{k-1},

$$\mathbb{P}_x(T > t) \sim \kappa \, \frac{\prod_{1\le i<j\le k}(x_i - x_j)}{t^{k(k-1)/4}}.$$

For D_k,

$$\mathbb{P}_x(T > t) \sim \kappa \, \frac{\prod_{1\le i<j\le k}(x_i^2 - x_j^2)}{t^{k(k-1)/2}}.$$

The constant κ is related to the Selberg-type integral

$$\Omega = \int_C e^{-|z|^2/2}\, h(z)\, dz.$$

In the example of A_{k-1}, (1.3) yields

$$\begin{aligned}\mathbb{P}_x(T > t) &= \int_C \det(p_t(x_i, y_j))_{1\le i,j\le k}\, dy \\ &= \frac{e^{-|x|^2/2t}}{(2\pi)^{k/2}} \int_C e^{-|z|^2/2} \det(e^{x_i z_j/\sqrt{t}})_{1\le i,j\le k}\, dz.\end{aligned}$$

Taking $x = \delta = (k-1, k-2, \dots, 0)$,

$$\det(e^{x_i z_j/\sqrt{t}})_{1\le i,j\le k} = h(z/\sqrt{t}) = h(z)/t^{n/2}$$

so that

$$\mathbb{P}_x(T > t) \sim \frac{t^{-n/2}}{(2\pi)^{k/2}}\Omega.$$

On the other hand, $h(\delta) = \prod_{j=1}^{k-1} j!$ from which it follows that $\Omega = (2\pi)^{-k/2} \prod_{j=1}^{k-1} j!\,\kappa$.

Similar formulae can be obtained in other cases relating the constant κ to the corresponding Selberg-type integral. General formulae for Selberg integrals associated with reflection groups are given by the Macdonald-Mehta conjectures (now proved) and can be found for example in [19].

4.6.2. Small time behaviour

Now the quantity of interest is $\mathbb{P}_x(T \le t)$ which goes to 0 as $t \to 0$. Such asymptotics for general domains are the subject of a vast literature in the setting of large deviations (see [10]). The interest of our direct approach is to provide very precise results when the domain is the chamber of some finite reflection group. Let us introduce notation $\theta(u) = 1 - \gamma(u) \sim \sqrt{\frac{2}{\pi}}u^{-1}e^{-u^2/2}$ as $u \to \infty$.

We will suppose that I is orthogonal and consistent, which only rules out B_k and F_4. Then,

$$\mathbb{P}_x(T \le t) = \sum_{B\in\mathcal{O}(\Pi)\backslash\{\emptyset\}} (-1)^{l(B)-1} \prod_{\alpha\in B} \theta(\hat{\alpha}(x)/\sqrt{t}). \tag{4.18}$$

The case of $I_2(m)$ with odd $m = 2m' + 1$ is particularly simple since $\mathcal{O}(\Pi) = \big\{\{\alpha_i\},\ 1 \le i \le m\big\}$ and $\mathbb{P}_x(T \le t) = \sum_i (-1)^{i-1}\theta\left((x,\alpha_i)/\sqrt{t}\right)$. Suppose $(x,\alpha_1) \le (x,\alpha_m)$. Then we have $(x,\alpha_1) \le (x,\alpha_m) < (x,\alpha_2) \le (x,\alpha_{m-1}) < (x,\alpha_3) \le (x,\alpha_{m-2}) < \cdots < (x,\alpha_{m'+1})$, which gives the hierarchy of terms in the asymptotic behaviour of $\mathbb{P}_x(T \le t)$ and enables the expansion with any given precision. For example, if x is fixed, $t \to \infty$ and noting $a_i = (x,\alpha_i)/\sqrt{t}$,

$$\mathbb{P}_x(T \le t) = \theta(a_1) + \theta(a_m) - \theta(a_2) - \theta(a_{m-1}) + o\left(\theta\left(\min(a_2, a_{m-1})\right)\right).$$

Similarly, one could deal with $I_2(m)$ for $m \equiv 2 \mod 4$.

Let us consider the general case and suppose there is only one root length, which is the case in all the examples where I is orthogonal. Although formula (4.18) allows to deal with general starting point, we will suppose for simplicity that x is at equal distance from all the walls of C, ie $\forall\alpha \in \Delta,\ d(x,\alpha^\perp) = (\alpha, x)/|\alpha| = c$. Then, $\hat{\alpha}(x) = c\,\mathrm{ht}(\alpha)$ where $\mathrm{ht}(\alpha) > 0$ is the sum of the coordinates of $\alpha \in \Pi$ in the basis Δ. Noting $a = c/\sqrt{t}$, the following equivalent hold:

$$q_B := \prod_{\alpha\in B} \theta(\hat{\alpha}(x)/\sqrt{t}) \sim c(B)\, a^{-|B|}\, e^{-n(B)a^2/2} \text{ as } a \to \infty,$$

where $n(B) = \sum_{\alpha\in B} \mathrm{ht}(\alpha)^2$ and $c(B) = \left(\frac{2}{\pi}\right)^{|B|/2} \left(\prod_{\alpha\in B} \mathrm{ht}(\alpha)\right)^{-1}$. Thus,

$$q_{B'} = o(q_B) \iff \big(n(B') > n(B) \text{ or } \big(n(B') = n(B) \text{ and } |B'| > |B|\big)\big).$$

Proposition 4.2. *The expansion up to some "order" n is given by*

$$\mathbb{P}_x(T \le t) = \sum_{n(B)\le n} (-1)^{l(B)-1} \prod_{\alpha\in B} \theta(ht(\alpha)a) + o\left(a^{-r}\, e^{-na^2/2}\right) \ \textit{as } a \to \infty, \tag{4.19}$$

where $r = \max\{|B| :\ n(B) = n\}$.

For instance, let us look at formula (4.19) for "small orders" $n = 1, 2, 3, 4$ in the crystallographic cases. Here they correspond to A_{k-1} and D_k. In this case, $\mathrm{ht}(\alpha) \in \mathbb{N}^*$ for all $\alpha \in \Pi$ so that $\mathrm{ht}(\alpha) = 1 \Leftrightarrow \alpha \in \Delta$. Recalling $n(B) = \sum_{\alpha\in B} \mathrm{ht}(\alpha)^2$, we see that

$$n(B) = i \Leftrightarrow (B \in \mathcal{O}(\Delta),\ |B| = i) \text{ for } i = 1, 2, 3.$$

Then

$$n(B) = 4 \Leftrightarrow (B \in \mathcal{O}(\Delta),\ |B| = 4) \text{ or } (B = \{\alpha\},\ \mathrm{ht}(\alpha) = 2).$$

The result can thus be written:

$$\mathbb{P}_x(T \le t) = \lambda_1\theta(a) - \lambda_2\theta(a)^2 + \lambda_3\theta(a)^3 - \lambda_4'\theta(2a) - \lambda_4\theta(a)^4 + o\left(a^{-4}e^{-4a^2/2}\right), \tag{4.20}$$

where $\lambda_i = |\{A \in \mathcal{O}(\Delta) :\ |A| = i\}|$ for $1 \le i \le 4$ and $\lambda_4' = \sum_{\mathrm{ht}(\alpha)=2} (-1)^{l(\{\alpha\})}$.

The case of A_{k-1} can be interpreted as k Brownian particles with equal distance c between consecutive neighbours. The constants in (4.20) can be explicitly computed: $\lambda_1 = k-1, \lambda_2 = (k-2)(k-3)/2$, $\lambda_3 = (k-5)(k-4)(k-3)/6$, $\lambda_4 = (k-7)(k-6)(k-5)(k-4)/24$ and $\lambda_4' = k-2$. This extends a previous result of [20] where the expansion was given up to "order" 2.

In the case of $D_k, \lambda_1 = k, \lambda_2 = (k-1)(k-2)/2, \lambda_3 = (k-4)(k-3)(k-2)/6+1$, $\lambda_4 = (k-6)(k-5)(k-4)(k-3)/24 + k - 5$ and $\lambda_4' = k-1$.

4.7. Expected exit times

4.7.1. The dihedral case

A well-known result of Spitzer [21] is that $\mathbb{E}_x(T^r) < \infty$ if and only if $m > r$, independently of x. Note that this also follows from the results of Section 4.6.1. In particular, $\mathbb{E}_x(T)$ is always finite. Let us concentrate on the case of odd m. It is impossible to get expectations directly by integrating (4.1) since the T_{α_i} are not integrable. The strategy is to use formula (4.1) to compute Laplace transforms:

$$\mathbb{E}_x(e^{-\lambda T}) = \sum_{i=1}^{m}(-1)^{i-1}\mathbb{E}_x(e^{-\lambda T_{\alpha_i}}) = \sum_{i=1}^{m}(-1)^{i-1}\exp(-(x,\alpha_i)\sqrt{2\lambda}),\ \lambda > 0. \tag{4.21}$$

By differentiation,

$$\mathbb{E}_x(Te^{-\lambda T}) = \frac{1}{\sqrt{2\lambda}}\left(\sum_{i=1}^{m}(-1)^{i-1}(x,\alpha_i)\exp(-(x,\alpha_i)\sqrt{2\lambda})\right).$$

Since $\sum_{i=1}^{m}(-1)^{i-1}(x,\alpha_i) = 0$ (cf Proposition 2.4), we can let $\lambda \to 0$ and get

$$\mathbb{E}_x(T) = \sum_{i=1}^{m}(-1)^i(x,\alpha_i)^2. \tag{4.22}$$

Remark 4.3. *In fact, identifying coefficients in the asymptotic expansion of (4.21) with respect to $\lambda \to 0$, we can express the moments:*

$$\mathbb{E}_x(T^r) = \frac{2^r\, r!}{(2r)!}\sum_{i=1}^{m}(-1)^{i+r-1}(x,\alpha_i)^{2r},\ 0 \le r \le m-1. \tag{4.23}$$

Writing $\cos^2\theta = \frac{1+\mathcal{R}(e^{i2\theta})}{2}$, the sum in (4.22) is computable using elementary geometric series: $\mathbb{E}_x(T) = \frac{r^2}{2}\left(\frac{\cos(2\theta-\pi/m)}{\cos(\pi/m)} - 1\right)$ for $x = re^{i\theta}$ with $r \ge 0$, $0 \le \theta \le \pi/m$. This formula makes good sense even if the angle α of the cone is not a fraction of π. It satisfies the Poisson equation $\Delta f = -2$ with the correct boundary conditions so that the exit time from the cone $C_\alpha = \{x = re^{i\theta} : r > 0,\ 0 < \theta < \alpha\}$ is:

$$\mathbb{E}_x(T) = \frac{r^2}{2}\left(\frac{\cos(2\theta-\alpha)}{\cos\alpha} - 1\right),\ x = re^{i\theta} \in C_\alpha. \tag{4.24}$$

Remark 4.4. *The formula (4.24) can be deduced from Spitzer's results. See, for example, Bramson and Griffeath [5].*

4.7.2. The A_{k-1} case

Since $\mathbb{P}_x(T > t) \sim C\, t^{-k(k-1)/4}$ as $t \to \infty$ (see Section 4.6.1), $\mathbb{E}_x(T^r) < \infty$ if and only if $r < k(k-1)/4$. In particular, $\mathbb{E}_x(T) = \infty$ for $k = 2$ (which is well-known) and $\mathbb{E}_x(T) < \infty$ for $k \geq 3$. Since A_2 is isomorphic to $I_2(3)$, it follows from the previous section that $\mathbb{E}_x(T) = (x_1 - x_2)(x_2 - x_3)$, which can be checked directly with the Poisson equation. Since $\gamma(a/\sqrt{t}) \sim C/\sqrt{t}$, formula (4.4) can only be integrated term by term if $k \geq 6$.

Let us first deal with the case $k = 4$ for which we use a kind of Laplace transform trick again and set:

$$E(\lambda) := \int_0^\infty e^{-\lambda t} \mathbb{P}_x(T > t)\, dt\ ,\ \lambda > 0, \tag{4.25}$$

where $\mathbb{P}_x(T > t) = \gamma(x_{12}/\sqrt{t})\gamma(x_{34}/\sqrt{t}) - \gamma(x_{13}/\sqrt{t})\gamma(x_{24}/\sqrt{t}) + \gamma(x_{14}/\sqrt{t})$ $\gamma(x_{23}/\sqrt{t})$ and $x_{ij} = x_i - x_j$. We write $\gamma(a/\sqrt{t}) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^a e^{-z^2/2t} \frac{dz}{\sqrt{t}}$ so that

$$\begin{aligned} J_\lambda(x_{ij}, x_{lm}) &:= \int_0^\infty e^{-\lambda t} \gamma(x_{ij}/\sqrt{t})\gamma(x_{lm}/\sqrt{t})\, dt \\ &= \frac{2}{\pi} \int_0^{x_{ij}}\!\!\int_0^{x_{lm}} \left(\int_0^\infty e^{-\lambda t - (z^2+w^2)/2t} \frac{dt}{t} \right) dz dw \end{aligned} \tag{4.26}$$

The integral between parenthesis can be written explicitly in terms of the Bessel function K_0 of the second kind (see, for example, [1, (29.3.119)])

$$J_\lambda(x_{ij}, x_{lm}) = \frac{4}{\pi} \int_0^{x_{ij}}\!\!\int_0^{x_{lm}} K_0\left(\sqrt{2\lambda(z^2 + w^2)}\right) dz dw. \tag{4.27}$$

We want to let $\lambda \to 0$ and use $K_0(x) = -\log(x/2) - c + \varepsilon(x)$, where c is the Euler constant and $\varepsilon(x) = o(x)$ as $x \to 0$ (see the Bessel function formula 9.6.13). Thus

$$J_\lambda(x_{ij}, x_{lm}) = \frac{4}{\pi} \left(x_{ij} x_{lm} c(\lambda) + \frac{1}{2} I(x_{ij}, x_{lm}) + r(\lambda) \right), \tag{4.28}$$

where $c(\lambda) = -\log(\sqrt{\lambda/2}) - c$, $I(a, b) := \frac{-1}{2} \int_0^a\!\int_0^b \log(z^2 + w^2)\, dz dw$ and $r(\lambda) \to 0$ as $\lambda \to 0$. Since $x_{12}x_{34} - x_{13}x_{24} + x_{23}x_{14} = 0$ (see Proposition 2.4), we obtain

$$E(\lambda) = \frac{2}{\pi} \left(I(x_{12}, x_{34}) - I(x_{13}, x_{24}) + I(x_{14}, x_{23}) \right) + r'(\lambda),$$

where $r'(\lambda) \to 0$ as $\lambda \to 0$. Letting $\lambda \to 0$ yields

$$\mathbb{E}_x(T) = F_2(x_1 - x_2, x_3 - x_4) - F_2(x_1 - x_3, x_2 - x_4) + F_2(x_1 - x_4, x_2 - x_3), \tag{4.29}$$

where we can explicitly compute

$$F_2(a,b)=\frac{2}{\pi}I(a,b)=\frac{2}{\pi}\Big(3ab+(a^2-b^2)\arctan(a/b)-\pi a^2/2-ab\log(a^2+b^2)\Big). \tag{4.30}$$

Remark 4.5. *We have $(\partial_a^2+\partial_b^2)F_2=-2$ from which we can check the Poisson equation for (4.29). The correct boundary conditions follow from $F_2(a,b)=F_2(b,a)$ and $F_2(0,b)=0$.*

The case $k=5$ is deduced from the previous one thanks to (4.7):

$$\mathbb{E}_x(T)=\sum_{l=1}^{5}(-1)^{l-1}\mathbb{E}_x(T_l), \tag{4.31}$$

where $\mathbb{E}_x(T_l)$ is the expression (4.29) applied to $(x_i)_{1\le i\ne l\le 5}$.

Let us now consider $k\ge 6$, note $p=\lfloor k/2\rfloor\ge 3$, use $\gamma(a/\sqrt{t})=\left(\frac{2}{\pi}\right)^{1/2}\int_0^a e^{-z^2/2t}\frac{dz}{\sqrt{t}}$ and integrate (4.4) term by term:

$$\mathbb{E}_x(T)=\sum_{\pi\in P_2(k)}(-1)^{c(\pi)}\left(\frac{2}{\pi}\right)^{p/2}\int_\pi\left(\int_0^\infty e^{-\sum z_{ij}^2/2t}\frac{dt}{t^{p/2}}\right)\prod_{\{i<j\}\in\pi}dz_{ij}, \tag{4.32}$$

where $\int_\pi$ denotes integration over $\prod_{\{i<j\}\in\pi}[0,x_{ij}]$. Now, we use

$$\int_0^\infty e^{-k/2t}\frac{dt}{t^q}=\frac{2^{q-1}\Gamma(q)}{k^{q-1}(q-1)},\ k>0,$$

and set

$$F_p(y_1,\dots,y_p)=\frac{2^{p+1}\Gamma(p/2)}{\pi^{p/2}(p-2)}\int_0^{y_1}\cdots\int_0^{y_p}\frac{dz_1\dots dz_p}{(z_1^2+\cdots+z_p^2)^{p/2-1}}, \tag{4.33}$$

to result in

$$\mathbb{E}_x(T)=\sum_{\pi\in P_2(k)}(-1)^{c(\pi)}F_p(x_\pi), \tag{4.34}$$

where $x_\pi=(x_i-x_j)_{\{i<j\}\in\pi}\in\mathbb{R}_+^p$. The previous result holds for $p=\lfloor k/2\rfloor\ge 2$.

Remark 4.6. *The formula (4.34) cannot be expressed as a Pfaffian since F_p does not have a product form.*

4.7.3. The D_k case

Since $\mathbb{P}_x(T > t) \sim C\, t^{-k(k-1)/2}$ as $t \to \infty$ (see Section 4.6.1), $\mathbb{E}_x(T^r) < \infty$ if and only if $r < k(k-1)/2$. In particular, $\mathbb{E}_x(T) = \infty$ for $k = 2$ and $\mathbb{E}_x(T) < \infty$ for $k \geq 3$. The method of computation is the same as for A_{k-1}, with the same obstacle for D_3 as for A_3. The general result for D_k is

$$\mathbb{E}_x(T) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} F_{2p}(x'_\pi), \tag{4.35}$$

where $p = \lfloor k/2 \rfloor \geq 2$, F_{2p} is defined in (4.30) an (4.33) and $x'_\pi = (x_i - x_j, x_i + x_j)_{\{i<j\} \in \pi} \in \mathbb{R}_+^{2p}$.

5. A generalisation of de Bruijn's formula

Suppose I is consistent. For $A \in \mathcal{I}$, denote dy W_A the group generated by reflections s_α, $\alpha \in A$. Denote by C_A the chamber associated to A, $C_A = \{x \in V : \forall \alpha \in A,\ (x, \alpha) > 0\}$. We will assume that C_A is a fundamental region for the reflection group W_A, which is certainly the case if I is orthogonal or semi-orthogonal.

Proposition 5.1. *If $f : V \to \mathbb{R}$ is integrable, then*

$$\int_C \sum_{w \in W} \varepsilon(w) f(wy)\, dy = \sum_{A \in \mathcal{I}} \varepsilon_A \sum_{w \in W_A} \varepsilon(w) \int_{C_A} f(wy)\, dy. \tag{5.1}$$

If I is orthogonal and $A = \{\alpha_1, \ldots, \alpha_l\} \in \mathcal{I}$, then $W_A \simeq \langle s_{\alpha_1} \rangle \times \cdots \times \langle s_{\alpha_l} \rangle$.

If I is semi-orthogonal and $A = \rho_1 \cup \ldots \cup \rho_l \in \mathcal{I}$, then $W_A \simeq \langle \rho_1 \rangle \times \cdots \times \langle \rho_l \rangle$, where each factor is either $\mathbb{Z}/2$ or $I_2(4)$ according whether ρ is a singlet or a pair.

5.1. The dihedral case

Recall that C is the wedge of angle π/m, $\alpha_j = e^{i(j\pi/m - \pi/2)}$, $\alpha'_j = e^{i(j\pi/m)}$, $E_j = \{x : (x, \alpha_j) > 0\}$, $E'_j = \{x : (x, \alpha'_j) > 0\}$ for $1 \leq j \leq m$.

Proposition 5.2. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is integrable, then*

$$\int_C \sum_{w \in W} \varepsilon(w) f(wy)\, dy$$
$$= \begin{cases} \sum_{j=1}^m (-1)^{j-1} \int_{E_j} \big(f(y) - f(s_j y)\big)\, dy & \textit{if } m \textit{ is odd} \\ \sum_{j=1}^m (-1)^{j-1} \int_{E_j \cap E'_j} \big(f(y) - f(s_j y) - f(-s_j y) + f(-y)\big)\, dy & \textit{if } m \equiv 2 \bmod 4\,. \end{cases}$$

5.2. Type A

Proposition 5.3. *If $f(y) = f_1(y_1) \ldots f_k(y_k)$ for integrable functions $f_i : \mathbb{R} \to \mathbb{R}$, then*

$$\int_C \det\big(f_i(y_j)\big)_{i,j \in [k]}\, dy_1 \ldots dy_k = Pf\,\big(I(f_i, f_j)\big)_{i,j \in [k]}\,, \ \textit{if } k \textit{ is even and,} \tag{5.2}$$

$$\int_C \det\left(f_i(y_j)\right)_{i,j\in[k]} dy_1 \dots dy_k = \sum_{l=1}^{k} (-1)^{l+1} \int_{\mathbb{R}} f_l \, Pf\left(I(f_i, f_j)\right)_{i,j\in[k]\setminus\{l\}}, \text{ if } k \text{ is odd}, \tag{5.3}$$

where I is the skew-symmetric bilinear form

$$I(f,g) = \int_{y>z} (f(y)g(z) - f(z)g(y)) \, dydz = \int sgn(y-z) f(y)g(z) \, dydz. \tag{5.4}$$

Remark 5.1. *This formula was first obtained by de Bruijn [6] using completely different methods. For a recent discussion with interesting connections to shuffle algebras, see [18].*

5.3. Type D

Proposition 5.4. *If $f(y) = f_1(y_1)\dots f_k(y_k)$ for integrable functions $f_i : \mathbb{R} \to \mathbb{R}$, then*

$$\int_C \det\left(f_i(y_j)\right)_{i,j\in[k]} dy_1 \dots dy_k = 0, \tag{5.5}$$

if the functions f_i are odd;

$$2^{(k-2)/2} \int_C \det\left(f_i(y_j)\right)_{i,j\in[k]} dy_1 \dots dy_k = Pf\left(K(f_i, f_j)\right)_{i,j\in[k]}, \tag{5.6}$$

if the functions f_i are even and k is even;

$$2^{(k-1)/2} \int_C \det\left(f_i(y_j)\right)_{i,j\in[k]} dy_1 \dots dy_k = \sum_{l=1}^{k} (-1)^{l+1} \int_{\mathbb{R}} f_l \, Pf\left(K(f_i, f_j)\right)_{i,j\in[k]\setminus\{l\}}, \tag{5.7}$$

if the functions f_i are even and k is odd; here, K is the skew-symmetric bilinear form

$$K(f,g) = \frac{1}{2} \int_{y>|z|} (f(y)g(z) - f(z)g(y)) \, dy \, dz. \tag{5.8}$$

Remark 5.2. *It is also possible to translate (5.1) into concrete terms for type B but the formula obtained is just a special case of Proposition 5.3.*

6. Random walks and related combinatorics

Let I_ν denote the Bessel function of index ν and recall that

$$I_\nu(x) = \sum_{l\geq 0} \frac{1}{l!\,\Gamma(\nu + l + 1)} (x/2)^{2l+\nu}.$$

Proposition 6.1. *Let $X_1, \dots, X_k$ be independent Poisson processes with unit rate and write $X = (X_1, \dots, X_k)$. Denote by T the first exit time of X from $C = \{x \in \mathbb{N}^k : x_1 > \cdots > x_k\}$. Then,*

$$\mathbb{P}_x(T > t) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\}\in\pi} q_{x_i - x_j}(t), \tag{6.1}$$

where $q_x(t) = \sum_{l=-x+1}^{x} p_l(t)$ and $p_y(t) = e^{-2t} I_{|y|}(2t)$.

Proposition 6.1 allows us to enumerate a certain class of Young tableaux. Let $\mathcal{T}_k(n)$ denote the set of standard Young tableaux with n boxes and height at most k and write $\tau_k(n)$ for the cardinality of $\mathcal{T}_k(n)$. If f^λ is the number of standard Young tableaux of shape λ, then $\tau_k(n) = \sum_{\lambda\vdash n,\, \lambda_1 \le k} f^\lambda$.

Proposition 6.2. *The generating function of $(\tau_k(n), n \ge 0)$ is given by*

$$y_k(t) := \sum_{n\ge 0} \frac{\tau_k(n)}{n!} t^n = \begin{cases} H_k\left(\gamma(t)\right) & \text{if } k \text{ is even,} \\ e^t\, H_k\left(\gamma(t)\right) & \text{if } k \text{ is odd,} \end{cases} \tag{6.2}$$

where $\gamma(t) = (\gamma_i(t))_{1\le i\le k-1}$, $\gamma_i(t) = \sum_{l=-i+1}^{i} I_{|i|}(t) = I_0(t) + 2\sum_{l=1}^{i-1} I_l(t) + I_i(t)$ and H_k is the polynomial

$$H_k(Y_1, \dots, Y_{k-1}) = \sum_{\pi\in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\}\in\pi} Y_{j-i} = Pf\left(Y_{j-i}\right),$$

with the convention $Y_l = -Y_{-l}$.

Proposition 6.3.

$$\tau_k(n) = \sum_{\lambda\vdash n, \lambda_1\le k} f_\lambda = \begin{cases} \sum_{\pi\in P_2(k)} (-1)^{c(\pi)} p_\pi(n) & \text{if } k \text{ is even;} \\ \sum_{\pi\in P_2(k)} (-1)^{c(\pi)} p'_\pi(n) & \text{if } k \text{ is odd,} \end{cases} \tag{6.3}$$

where, for $\pi = \{\{i_1, i_2\}, \dots, \{i_{k-1}, i_k\}\}$, $p_\pi(n)$ (respectively $p'_\pi(n)$) is the number of lattice paths of length n (respectively at most n) in $\mathbb{N}^{k/2}$ started at $x_\pi := (|i_1 - i_2|, \dots, |i_{k-1} - i_k|)$.

The formula (6.2) is related to a formula obtained by Gordon [14] concerning plane partitions. Following [22] we can transform this Pfaffian expression into a determinantal expression to recover the following well-known identity of Gessel [11]:

$$y_k(t) = \begin{cases} \det[I_{i-j}(2x) + I_{i+j-1}(2x)]_{1\le i,j\le k/2} & \text{if } k \text{ is even;} \\ e^t\, \det[I_{i-j}(2x) - I_{i+j}(2x)]_{1\le i,j\le (k-1)/2} & \text{if } k \text{ is odd.} \end{cases}$$

Similar formulas exist for $\sum_{\lambda\vdash n,\, \lambda_1\le k} s_\lambda$ (see, for example, [22]); these can also be obtained by our approach if one considers a slightly different class of random walks.

7. Proofs

7.1. The main result

Recall that, if $\alpha \in \Delta$, then $s_\alpha(\Pi\backslash\{\alpha\}) = \Pi\backslash\{\alpha\}$. (See, for example, [16, Proposition 1.4].) We begin with the following key lemma.

Lemma 7.1. *If $K \subset I$ and $\alpha \in \Delta \cap K^\perp$, then $s_\alpha L = L$, where*

$$L = \{w \in W^I : \; K \subset wI, \; \alpha \notin wI\}.$$

Proof. Suppose $w \in L$. α being the only positive root made negative by s_α, ($wI \subset \Pi$ and $\alpha \notin wI$) implies that $s_\alpha wI \subset \Pi$. If $\alpha \in s_\alpha wI$ then $s_\alpha(\alpha) = -\alpha \in wI$, which is absurd since $wI \subset \Pi$. Hence $\alpha \notin s_\alpha wI$. Seeing that $K \subset wI$, we get $K = s_\alpha K \subset s_\alpha wI$. This proves that $s_\alpha w \in L$. Thus $s_\alpha L \subset L$. Applying s_α to the previous inclusion, we get $L \subset s_\alpha L$, whence the equality. □

Lemma 7.2. *If $g : W^I \to \mathbb{R}$ and $\alpha \in \Delta$ are such that $g(w) = 0$ whenever $\alpha \in wI$, and $g(w) = g(s_\alpha w)$ whenever $\alpha \notin wI$, then $\sum_{w \in W^I} \varepsilon(w) g(w) = 0$.*

Proof. Applying lemma 7.1 with $K = \phi$,

$$\sum_{w \in W^I} \varepsilon(w)g(w) = \sum_{w \in W^I, \alpha \notin wI} \varepsilon(w)g(w) = \sum_{w \in W^I, \alpha \notin wI} \varepsilon(s_\alpha w)g(s_\alpha w)$$
$$= -\sum_{w \in W^I, \alpha \notin wI} \varepsilon(w)g(w),$$

which must therefore be zero. □

Lemma 7.3. *If condition (C1) is satisfied, then*

$$\sum_{w \in W^I} \varepsilon(w) = \sum_{w \in U} \varepsilon(w).$$

Proof. By lemma 7.1, for any $K \subset I$ and $\alpha \in \Delta \cap K^\perp$,

$$\sum_{w \in W^I, K \subset wI, \alpha \notin wI} \varepsilon(w) = \sum_{w \in W^I, K \subset wI, \alpha \notin wI} \varepsilon(s_\alpha w) = -\sum_{w \in W^I, K \subset wI, \alpha \notin wI} \varepsilon(w)$$

so that $\sum_{w \in W^I, K \subset wI, \alpha \notin wI} \varepsilon(w) = 0$ and

$$\sum_{w \in W^I, K \subset wI} \varepsilon(w) = \sum_{w \in W^I, K \cup \{\alpha\} \subset wI} \varepsilon(w). \tag{7.1}$$

If $J = \{\alpha_1, \ldots, \alpha_q\}$, we apply (7.1) successively to $(K, \alpha) = (\emptyset, \alpha_1)$, $(K, \alpha) = (\{\alpha_1\}, \alpha_2)$, etc, until exhaustion of J. We then obtain:

$$\sum_{w \in W^I} \varepsilon(w) = \sum_{w \in W^I, J \subset wI} \varepsilon(w).$$

Now, hypothesis (C1) makes sure that the last sum runs over those w such that $wI = I$, that is $w \in U$. Thus, the result is proved. □

Proposition 7.1. *Suppose I is consistent. Then:*

(i) $\sum_{A\in\mathcal{I}} \varepsilon_A = 1$.
(ii) *If $f : \mathcal{I} \to \mathbb{R}$ and $\alpha \in \Delta$ are such that $f(A) = 0$ whenever $\alpha \in A$, and $f(A) = f(s_\alpha A)$ whenever $\alpha \notin A$, then $\sum_{A\in\mathcal{I}} \varepsilon_A f(A) = 0$.*

Proof. If $f : \mathcal{I} \to \mathbb{R}$ then $\sum_{w\in W^I} \varepsilon(w) f(wI) = \sum_{A\in\mathcal{I}} \sum_{wI=A} \varepsilon(w) f(wI) = |U| \sum_{A\in\mathcal{I}} \varepsilon_A f(A)$ since $|\{w : wI = A\}| = |U|$ for every $A \in \mathcal{I}$. Hence (ii) follows from lemma 7.2 and (i) follows from lemma 7.3 and the previous computation with $f = 1$. □

Proof of proposition 2.3. We check the boundary conditions for $u(x,t) = \sum_{A\in\mathcal{I}} \varepsilon_A \mathbb{P}_x(T_A > t)$.
If $x \in C$, the regularity hypothesis (2.4) guarantees that $\mathbb{P}_x(T_A > 0) = 1$ for all A, which implies $u(x,0) = 1$ thanks to (i) of Proposition 7.1.
Suppose $x \in \partial C$, choose $\alpha \in \Delta$ such that $(\alpha, x) = 0$ and set $f(A) = \mathbb{P}_x(T_A > t)$. If $\alpha \in A$ then $f(A) = 0$. If $\alpha \notin A$ then use invariance of the law of X under s_α and the fact that $s_\alpha x = x$ to get $f(s_\alpha A) = f(A)$. $u(x,t) = 0$ follows from (ii) of Proposition 7.1. □

7.2. Bijection and cancellation lemmas

Lemma 7.4. *Suppose $\alpha \in \Delta$.*

(i) *$\theta : A \to s_\alpha A$ is a permutation of $\{A \in \mathcal{I} : \alpha \notin A\}$.*
(ii) *Suppose I is orthogonal. If $B \in \mathcal{O}(\Pi)$ and $\alpha \notin B^\perp$ then $\Gamma : A \to s_\alpha A$ is a bijection from $F = \{A \in \mathcal{I} : B \subset s_\alpha A\}$ to $G = \{A \in \mathcal{I} : B \subset A\}$.*

Proof. (i) follows from lemma 7.1 with $K = \emptyset$.
For (ii), suppose $A \in F$ and $s_\alpha A \not\subseteq \Pi$. Then $\alpha \in A$, so that $s_\alpha A = \{-\alpha\} \cup (A\backslash\{\alpha\})$ and, since $-\alpha \notin B$, $B \subset A\backslash\{\alpha\}$. This implies $\alpha \in B^\perp$, which is absurd. Thus, Γ is well-defined. In the same way, $A \to s_\alpha A$ is well-defined from G to F, which proves (ii). □

Lemma 7.5. *Suppose I is orthogonal. If a function $G(x, A)$, defined for $x \in V$ and $A \subset\in \mathcal{I}$, satisfies:*

(i) *$G(s_\alpha x, A) = G(x, s_\alpha A)$ for $\alpha \in \Phi$,*
(ii) *$G(x, \{-\alpha\} \cup (A\backslash\{\alpha\})) = -G(x, A)$ for $\alpha \in A$,*

then $F(x) = \sum_{A\in\mathcal{I}} \varepsilon_A\, G(x, A)$ satisfies $F(wx) = \varepsilon(w)F(x)$ for $w \in W$. In particular, if F is a polynomial in x then $h(x) = \prod_{\alpha\in\Pi}(x, \alpha)$ divides $F(x)$. If I is only semi-orthogonal, the same result holds if one adds to (ii) *the following property: $G(x, s_\alpha\{\alpha, \beta\} \cup (A\backslash\{\alpha, \beta\}) = -G(x, A)$ for $\{\alpha, \beta\} \in A^*$.*

Proof. The proof is done for the orthogonal case but is the same in the semi-orthogonal one, modulo obvious modification. Use successively hypotheses (i), (ii), $\varepsilon_{s_\alpha A} = -\varepsilon_A$ and the bijection given by (i) of lemma 7.4 to compute

$$
\begin{aligned}
F(s_\alpha x) &= \sum_{A\in\mathcal{I}} \varepsilon_A\, G(s_\alpha x, A) = \sum_{A\in\mathcal{I}} \varepsilon_A\, G(x, s_\alpha A)\\
&= \sum_{A\in\mathcal{I},\,\alpha\in A} \varepsilon_A\, G(x, \{-\alpha\}\cup(A\backslash\{\alpha\})) - \sum_{A\in\mathcal{I},\,\alpha\notin A} \varepsilon_{s_\alpha A}\, G(x, s_\alpha A)\\
&= -\sum_{A\in\mathcal{I},\,\alpha\in A} \varepsilon_A\, G(x, A) - \sum_{A\in\mathcal{I},\,\alpha\notin A} \varepsilon_A\, G(x, A)\\
&= -F(x)
\end{aligned}
$$

□

Proof of proposition 2.4. Define $G(x, A) = \prod_{\alpha\in A}(x, \alpha)$ which satisfies hypotheses of lemma 7.5. Then $F(x) = P(x)$ is a polynomial divisible by $h(x) = \prod_{\alpha\in\Pi}(x, \alpha)$. Since $\deg P = |I| < |\Pi| = \deg h$, we have $P = 0$. □

Proof of proposition 2.8. Define

$$
G(x, A) = \prod_{\{\alpha,\beta\}\in A^*} (\alpha, x)(\beta, x)(s_\alpha\beta, x)(s_\beta\alpha, x) \prod_{\gamma\in A}(x, \gamma)
$$

which satisfies hypotheses of lemma 7.5. Then $F(x) = P(x)$ is a polynomial divisible by $h(x) = \prod_{\alpha\in\Pi}(x, \alpha)$. Since $\deg P \le 2|I| < |\Pi| = \deg h$, we have $P = 0$. □

7.3. *The dual formula*

Proof of proposition 2.5. Let $B = \{\beta_1, \dots, \beta_k\} \in \mathcal{O}(\Pi)$. It suffices to show that $B = \alpha_j.\cdots.\alpha_1.B'$ for some sequence $\alpha_1, \dots, \alpha_j \in \Delta$ and $B' \in \mathcal{O}(\Pi)$ with $|B'| < |B|$. If B contains a simple root α we are done because then $B = \alpha.(B\backslash\{\alpha\})$. Assume that $B \cap \Delta = \phi$. We can write $\beta_1 = s_{\alpha_j}\cdots s_{\alpha_2}\alpha_1$ for some sequence $\alpha_1, \dots, \alpha_j \in \Delta$ such that α_{i+1} is neither equal or orthogonal to $s_{\alpha_i}\cdots s_{\alpha_2}\alpha_1$ for all $i = 1, \dots, j-1$. Now set $B' = (s_{\alpha_2}\cdots s_{\alpha_j}B)\backslash\{\alpha_1\}$. Then $B' \in \mathcal{O}(\Pi)$ and $B = \alpha_j.\cdots.\alpha_1.B'$, as required. To see this, set $B_1 = \alpha_1.B' = s_{\alpha_2}\cdots s_{\alpha_j}B$ and $B_{i+1} = s_{\alpha_{i+1}}B_i$ for $i = 1, \dots, j-1$. Note that $\alpha_1 \in B_1$ and $s_{\alpha_i}\cdots s_{\alpha_2}\alpha_1 \in B_i$ for $i = 2, \dots, j-1$. Since B_i is orthogonal, and α_{i+1} is neither equal to, or orthogonal to, $s_{\alpha_i}\cdots s_{\alpha_2}\alpha_1$, it follows that $\alpha_{i+1} \notin B_i$ and hence $s_{\alpha_{i+1}}B_i = \alpha_{i+1}.B_i$ for each $i = 1, \dots, j-1$ and, moreover, that B' contains only positive roots. □

Proof of proposition 2.6. Since $\sum_{A\in\mathcal{I}}\varepsilon_A = 1$, we can write

$$
\mathbb{P}_x[T \le t] = \sum_{A\in\mathcal{I}} \varepsilon_A \mathbb{P}_x[T_A \le t].
$$

By the inclusion-exclusion principle,

$$
\mathbb{P}_x[T_A \le t] = \sum_{B\subset A} (-1)^{|B|-1}\mathbb{P}_x[\forall\beta\in B,\ T_\beta \le t].
$$

Thus

$$\mathbb{P}_x[T \le t] = \sum_{B \in \mathcal{O}(\Pi)} \nu_B \mathbb{P}_x[\forall \beta \in B,\ T_\beta \le t],$$

where $\nu_B = (-1)^{|B|-1} \sum_{A \in \mathcal{I},\, B \subset A} \varepsilon_A$. We will prove that $\nu_B = (-1)^{l(B)-1}$ by induction on $l(B)$. The result for $l(B) = 0$ is just (i) of proposition 7.1.
Suppose that $l(B) = l \ge 1$, write $B = \alpha_l \ldots \alpha_2.\alpha_1.\emptyset$. Set $\alpha = \alpha_l$ and $B' = \alpha_{l-1} \ldots \alpha_1.\emptyset$ so that $B = \alpha.B'$. We have $l(B') = l - 1$ and $\alpha \notin B'$ (the contrary would contradict $l(B) = l$).
The first case is $\alpha \in B'^{\perp}$. Then $|B'| = l - 1$ and $B = B' \cup \{\alpha\}$. Thus,

$$\begin{aligned} \nu_B &= (-1)^{|B|-1} \sum_{A \in \mathcal{I},\, \alpha \in A,\, B' \subset A} \varepsilon_A \\ &= (-1)^{|B|-1} \Big(\sum_{A \in \mathcal{I},\, B' \subset A} \varepsilon_A - \sum_{A \in \mathcal{I}} \varepsilon_A f(A) \Big), \end{aligned}$$

where $f(A) = \mathbf{1}_{\alpha \notin A,\, B' \subset A}$. If $\alpha \in A$, then $f(A) = 0$. If $\alpha \notin A$, then $f(A) = \mathbf{1}_{B' \subset A}$. Using $\alpha \notin s_\alpha A$ and $s_\alpha B' = B'$, we obtain $f(s_\alpha A) = \mathbf{1}_{B' \subset s_\alpha A} = \mathbf{1}_{s_\alpha B' \subset A} = f(A)$. Again, (ii) of proposition 7.1 applies and $\nu_B = (-1)^{|B|-1} \sum_{A \in \mathcal{I},\, B' \subset A} \varepsilon_A = -\nu_{B'}$, which concludes this case.
The second case is $\alpha \notin B'^{\perp}$. Then $|B'| = l$, $B = s_{\alpha_l} B'$ and

$$\nu_B = (-1)^{|B|-1} \sum_{A \in \mathcal{I},\, s_\alpha B' \subset A} \varepsilon_A = (-1)^{|B|} \sum_{A \in \mathcal{I},\, B' \subset s_\alpha A} \varepsilon_{s_\alpha A}. \tag{7.2}$$

Thanks to (ii) of lemma 7.4, $A \mapsto s_\alpha A$ is a bijection from $\{A \in \mathcal{I} :\ B' \subset s_\alpha A\}$ to $\{A' \in \mathcal{I} :\ B' \subset A'\}$, so that (7.2) transforms into $\nu_B = (-1)^{|B|} \sum_{A' \in \mathcal{I},\, B' \subset A'} \varepsilon_{A'} = -\nu_{B'}$ and we are done. □

7.4. Consistency

7.4.1. The dihedral groups

For m odd, we take $J = \{\alpha\}$. The angle between two roots being a multiple of π/m, no two roots are orthogonal. Therefore, the unique extension property of Lemma 3.1 is trivially verified with $I = J$. Then, $U = \{\mathrm{id}\}$, which guarantees condition (C2). Thus, I is consistent and $\mathcal{I} = \{\{\alpha_1\}, \ldots, \{\alpha_m\}\}$ with $\varepsilon_{\{\alpha_i\}} = (-1)^{i-1}$.

For $m = 2m'$ with m' odd, we again choose $J = \{\alpha\}$. Now, the unique extension property of Lemma 3.1 is verified with $J = \{\alpha\}$, $I = \{\alpha, \alpha' = e^{i\pi/2}\alpha\}$. Suppose w leaves I invariant. Since I is a basis of $\mathbb{R}^2$, if w fixes I pointwise, then $w = \mathrm{id}$. Otherwise, w permutes α and α'. Then, w has to be the reflection with respect to the bisecting line of α and α'. Since $\pi/4$ is not a multiple of π/m, the latter reflection is not in W. Hence, $U = \{\mathrm{id}\}$ and (C2) is satisfied. Writing $\alpha_i' = e^{i\pi/2}(\alpha_i)$, I is consistent again with $\mathcal{I} = \big\{\{\alpha_1, \alpha_1'\}, \ldots, \{\alpha_m, \alpha_m'\}\big\}$ and $\varepsilon_{\{\alpha_i, \alpha_i'\}} = (-1)^{i-1}$.

7.4.2. A_{k-1}

We define $p = \lfloor k/2 \rfloor$. The choice of $J = I$ makes Lemma 3.1 trivially verified. The following lemma is obvious:

Lemma 7.6. *We can characterize the sets W^I and U:*

$$\begin{aligned} W^I &= \{\sigma \in \mathfrak{S}_k : \forall i \in [p],\ \sigma(2i-1) < \sigma(2i)\}, \\ U &= \{\sigma \in \mathfrak{S}_k : \sigma \text{ permutes the consecutive pairs } (1,2), (3,4), \ldots, (2p-1, 2p)\} \\ &= \{\sigma \in \mathfrak{S}_k : \exists \tau \in \mathfrak{S}_p,\ \forall i \in [p],\ \sigma(2i-1) = 2\tau(i) - 1,\ \sigma(2i) = 2\tau(i)\}. \end{aligned}$$

Proof of proposition 3.1. We will use the sign function: $s(x) = 1$ if $x \geq 0$ and $s(x) = -1$ if $x < 0$. Let us first suppose that k is even. The sign of $\sigma \in \mathfrak{S}_k$ can be expressed as

$$\varepsilon(\sigma) = \prod_{j<i} s\,(\sigma(i) - \sigma(j)) = P_{00}\, P_{11}\, P_{01}\, P_{10}\, Q, \tag{7.3}$$

where

$$P_{ab} = \prod_{1 \leq m < l \leq p} s\,(\sigma(2m - a) - \sigma(2l - b)) \quad \text{for } a, b \in \{0, 1\}$$

and

$$Q = \prod_{1 \leq l \leq p} s\,(\sigma(2l) - \sigma(2l-1))\,.$$

If $\sigma \in U$, using notation of lemma 7.6,

$$P_{00} P_{11} = \prod_{1 \leq m < l \leq p} s\,(\tau(l) - \tau(m))^2 = 1,$$

$$P_{01} P_{10} = \prod_{1 \leq m < l \leq p} s\left(4(\tau(l) - \tau(m))^2 - 1\right) = 1$$

and $Q = 1$. So (i) is proved.

For (ii), we note that

$$\sigma I = \{e_{\sigma(1)} - e_{\sigma(2)}, e_{\sigma(3)} - e_{\sigma(4)}, \ldots, e_{\sigma(k-1)} - e_{\sigma(k)}\}$$

and we define the mapping θ from $\mathcal{I}$ to $P_2(k)$ by

$$\theta(\sigma I) = \Big\{\{\sigma(1), \sigma(2)\}, \{\sigma(3), \sigma(4)\}, \ldots, \{\sigma(k-1), \sigma(k)\}\Big\}.$$

This is well-defined since, if $\sigma I = \sigma' I$, then $\sigma = \sigma' \tau$ with $\tau \in U$, so

$$\{\sigma(2i-1), \sigma(2i)\} = \{\sigma'(\tau(2i-1)), \sigma'(\tau(2i))\}.$$

Since τ permutes consecutive pairs $(2i-1, 2i)$, $\theta(\sigma I)$ and $\theta(\sigma' I)$ are the same partition. θ is obviously injective and onto.

For (iii), notice that if $\pi = \{\{j_1, j_1'\}, \ldots, \{j_p, j_p'\}\}$ with $j_i < j_i'$ then

$$(-1)^{c(\pi)} = \prod_{1 \le m < l \le p} s\left((j_l - j_m)(j_l - j_m')(j_l' - j_m)(j_l' - j_m')\right). \tag{7.4}$$

Construct σ by $\sigma(2i-1) = j_i, \sigma(2i) = j_i'$. Then $\theta(\sigma I) = B$ and comparing (7.4) with (7.3) yields $(-1)^{c(\pi)} = \varepsilon(\sigma)$.

If k is odd and $\sigma \in U$, then $\sigma(k) = k$ so that formula (7.3) is still true and the same proof holds for (i). θ is given by

$$\theta(\sigma I) = \left\{\{\sigma(1), \sigma(2)\}, \{\sigma(3), \sigma(4)\}, \ldots, \{\sigma(k-2), \sigma(k-1)\}, \{\sigma(k)\}\right\}.$$

As for (iii), suppose $\sigma \in W^I$, $\theta(\sigma I) = \pi$ and define π' to be the partition π deprived of its singlet $\{\sigma(k)\}$. Then formula (7.3) must be modified into

$$\varepsilon(\sigma) = P_{00}\, P_{11}\, P_{01}\, P_{10}\, Q \prod_{j=1}^{k-1} (\sigma(k) - \sigma(j)).$$

The first part equals $(-1)^{c(\pi')}$. The second is $(-1)^{\sigma(k)+1}$, which exactly corresponds with the number of crossings obtained by adding the pair $\{0, \sigma(k)\}$ with 0 at the left of everything. $\square$

Proof of remark 3.1. We prove the formula $i(\sigma) = c(\pi) + 2b(\pi)$. Recall that

$$\pi = \left\{\{j_1, j_1'\}, \{j_2, j_2'\}, \ldots, \{j_p, j_p'\}\right\} \in P_2(k), \quad j_i < j_i', \quad j_1 < j_2 < \cdots < j_p.$$

Construct σ by $\sigma(2i-1) = j_i, \sigma(2i) = j_i'$. We define the set of crossings of π,

$$\mathrm{Cr} = \{(i, l) \in [p]^2 : \mathrm{i} < l,\ j_i < j_l < j_i' < j_l'\},$$

and the set of bridges of π,

$$\mathrm{Br} = \{(i, l) \in [p]^2 :\ i < l,\ j_i < j_l < j_l' < j_i'\}.$$

If $(i, l) \in \mathrm{Cr}$, then $\sigma(2l-1) < \sigma(2i)$ and $2i < 2l - 1$. Hence we set $\phi(i, l) = \{(2i, 2l-1)\} \subset \mathrm{Inv}(\sigma)$. If $(i, l) \in \mathrm{Br}$, then $\sigma(2l-1) < \sigma(2l) < \sigma(2i)$ and $2i < 2l - 1$. Hence we set $\psi(i, l) = \{(2i, 2l-1), (2i, 2l)\} \subset \mathrm{Inv}(\sigma)$.

We claim that:

$$\Big(\bigcup_{(i,l)\in \mathrm{Cr}} \phi(i, l)\Big) \bigcup \Big(\bigcup_{(i,l)\in \mathrm{Br}} \psi(i, l)\Big) = \mathrm{Inv}(\sigma). \tag{7.5}$$

Since $\mathrm{Cr} \cap \mathrm{Br} = \emptyset$, the union in (7.5) is disjoint and (3.4) follows. For (7.5), we have already proved the inclusion of the left-hand side in $\mathrm{Inv}(\sigma)$.

Conversely, take $(a, b) \in \mathrm{Inv}(\sigma)$. Suppose a is odd, $a = 2i - 1$, then $\sigma(a) = j_i$ and for $l > i$, we have $j_l' > j_l > j_i' > j_i$, i.e. $\sigma(2l) > \sigma(2l-1) > \sigma(2i) > \sigma(2i-1)$. This means that $(a, b) \notin \mathrm{Inv}(\sigma)$. Hence a is even, $a = 2i$.

If b is even, $b = 2l$, then $i < l$. Hence $\sigma(2i-1) = j_i < \sigma(2l-1) = j_l < \sigma(2l) = j'_l < \sigma(2i) = j'_i$, the first two inequalities being trivial and the last one due to $(a,b) \in \mathrm{Inv}(\sigma)$. Thus $(a,b) \in \psi(i,l)$ and $(i,l) \in \mathrm{Br}$.

If b is odd, $b = 2l-1$, $2i < 2l-1$ and $\sigma(2i-1) = j_i < \sigma(2l-1) = j_l < \sigma(2i) = j'_i$. Two possibilities arise: either $j_l < j'_l < j'_i$, then $(a,b) \in \psi(i,l)$ and $(i,l) \in \mathrm{Br}$, or $j'_i < j'_l$, then $(a,b) \in \phi(i,l)$ and $(i,l) \in \mathrm{Cr}$.

Anyhow, $(a,b) \in \left(\bigcup_{(i,l)\in\mathrm{Cr}} \phi(i,l)\right) \bigcup \left(\bigcup_{(i,l)\in\mathrm{Br}} \psi(i,l)\right)$, which ends the proof. □

7.4.3. D_k

We define $p = \lfloor k/2 \rfloor$. By choosing $J = \{e_1 - e_2, e_3 - e_4, \dots, e_{k-1} - e_k, e_{k-1} + e_k\}$ if k is even and $J = \{e_2 - e_3, e_4 - e_5, \dots, e_{k-1} - e_k, e_{k-1} + e_k\}$ otherwise, the unique extension property of Lemma 3.1 is easily checked.

Lemma 7.7. *We can characterize the sets W^I and U:*

$$W^I = \{f\sigma \in W : \forall i \in [p],\ \sigma(2i-1) \notin \overline{f},\ \sigma(2i-1) < \sigma(2i)\} \text{ if } k \text{ is even},$$
$$W^I = \{f\sigma \in W : \forall i \in [p],\ \sigma(2i) \notin \overline{f},\ \sigma(2i) < \sigma(2i+1)\} \text{ if } k \text{ is odd},$$
$$U = \{f\sigma \in W : \sigma \text{ permutes the consecutive pairs } (1,2), (3,4), \dots, (k-1,k);$$
$$\forall i \in [p],\ \sigma(2i-1) \notin \overline{f}\} \text{ if } k \text{ is even},$$
$$U = \{f\sigma \in W : \sigma \text{ permutes the consecutive pairs } (2,3), (4,5), \dots, (k-1,k);$$
$$\forall i \in [p],\ \sigma(2i) \notin \overline{f}\} \text{ if } k \text{ is odd}.$$

Proof. Let us do proofs for even k. Suppose $f\sigma \in W^I$ then $f\sigma(e_{2i-1} \pm e_{2i}) = f(e_{\sigma(2i-1)}) \pm f(e_{\sigma(2i)}) \in \Pi$, which implies $\sigma(2i-1) \notin \overline{f}$ and then $\sigma(2i) > \sigma(2i-1)$. This proves the first equality. The second one is then obvious. □

Proof of proposition 3.1 for D_k. The identification between $\mathcal{I}$ and $P_2(k)$ is done via θ which sends $f\sigma I = \{e_{\sigma(2i-1)} \pm e_{\sigma(2i)},\ i \in [p]\}$ to:

$$\begin{cases} \left\{\{\sigma(2i-1), \sigma(2i)\},\ i \in [p]\right\} & \text{if } k \text{ is even}, \\ \left\{\{\sigma(1)\}; \{\sigma(2i-1), \sigma(2i)\},\ i \in [p]\right\} & \text{if } k \text{ is odd}. \end{cases}$$

Since $\varepsilon(f) = 1$ for $f\sigma \in W$, (C2) and $\varepsilon_A = (-1)^{c(\theta(A))}$ immediately come from the analogous facts for A_{k-1}. □

7.4.4. B_k

We define $p = \lfloor k/2 \rfloor$. Our choice is $J = \{e_1 - e_2, e_3 - e_4, \dots, e_{k-1} - e_k\}$ if k is even and $J = \{e_1 - e_2, e_3 - e_4, \dots, e_{k-2} - e_{k-1}, e_k\}$ if k is odd. Then Lemma 3.1 is easily verified.

Lemma 7.8. *We can characterize the sets W^I and U:*

$$W^I = \{\sigma \in \mathfrak{S}_k : \forall i \in [p],\ \sigma(2i-1) < \sigma(2i)\},$$
$$U = \{\sigma \in \mathfrak{S}_k : \sigma \text{ permutes the consecutive pairs } (1,2), (3,4), \dots, (2p-1, 2p)\}.$$

Proof. Suppose $f\sigma \in W^I$ then (1) : $f\sigma(e_{2i-1}-e_{2i}) = f(e_{\sigma(2i-1)}) - f(e_{\sigma(2i)}) \in \Pi$ and (2) $f\sigma(e_{2i}) = f(e_{\sigma(2i)}) \in \Pi$. (1) implies $\sigma(2i-1) \notin \overline{f}$ and then (2) yields $\sigma(2i) \notin \overline{f}$. So $f = \text{id}$ and consequently $\sigma(2i) > \sigma(2i-1)$. This proves the first equality. The second and third ones are then obvious. □

Proof of proposition 3.1 for B_k. The proof of (C2) is the same as for A_{k-1}. The identification between $\mathcal{I}$ and $P_2(k)$ is done via θ. If k is even and $A = \sigma I = \{e_{\sigma(2i-1)} - e_{\sigma(2i)}, e_{\sigma(2i)},\ i \in [p]\}$ then

$$\theta(A) = \big\{\{\sigma(2i-1), \sigma(2i)\},\ i \in [p]\big\}.$$

If k is odd and $A = \sigma I = \{e_{\sigma(2i-1)} - e_{\sigma(2i)}, e_{\sigma(2i)},\ i \in [p];\ e_{\sigma(k)}\}$, then

$$\theta(A) = \big\{\{\sigma(2i-1), \sigma(2i)\},\ i \in [p];\ \{\sigma(k)\}\big\}.$$

Then, $\varepsilon_A = (-1)^{c(\theta(A))}$ directly comes from the A_{k-1} case. □

7.4.5. H_3 and H_4

Let us first deal with H_3, take $J = \{\alpha = a - i/2 + bj, \beta = 1/2 + bi - aj\}$, define $\gamma = b + ai + j/2$. Then, $I = \{\alpha, \beta, \gamma\}$. Lemma 3.1 is trivially verified since the linear span of Φ is three-dimensional. Now, suppose $\varepsilon(w) = -1$ and $wI = I$. w acts as an odd permutation of $\{\alpha, \beta, \gamma\}$ so, for example, w is the transposition (α, β). Thus, $w(\alpha + \beta) = \alpha + \beta$, $w\gamma = \gamma$ and $w(\alpha - \beta) = -\alpha + \beta$, which means that $w = s_{\alpha-\beta}$. This is absurd since $\mathbb{R}(\alpha - \beta)$ contains no root of H_3. The same being true for $\mathbb{R}(\alpha - \gamma)$ and $\mathbb{R}(\beta - \gamma)$, the proof is done.

For H_4, take $J = \{\alpha = -a + i/2 + bj, \beta = -1/2 - ai + bk\}$, define $\gamma = b + aj + k/2$, $\delta = bi - j/2 + ak$. Then, $I = \{\alpha, \beta, \gamma, \delta\}$ satisfies lemma 3.1 since $\mathbb{R}^4 \equiv \mathbb{H}$ is four-dimensional. Suppose $\varepsilon(w) = -1$ and $wI = I$. w acts as an odd permutation of $\{\alpha, \beta, \gamma, \delta\}$, so is either a transposition or a 4-cycle. A transposition is ruled out by the same analysis as H_3, since the differences between elements of I are not multiples of roots. If w is the 4-cycle $(\alpha\beta\gamma\delta)$ then consider the root $\lambda = (1 + i + j + k)/2$. Since $(\lambda, \alpha) = (\gamma, i) = 0$, we have $(w\lambda, i) = (\lambda, \gamma)(\delta, i) + (\lambda, \delta)(\alpha, i) = (2a + 2b + 1)/8 \notin \{(\phi, i);\ \phi \in \Phi\}$. Hence, $w\lambda \notin \Phi$, which proves that $w \notin W$.

7.4.6. F_4

Take $J = \{e_2 - e_3, e_4\}$ and $I = \{e_2 - e_3, e_3, e_1 - e_4, e_4\}$. I is semi-orthogonal and lemma 3.1 is an easy check. Suppose $wI = I$. Since w sends long roots to long roots and short roots to short roots, $w \in \mathfrak{S}_{\{e_3, e_4\}} \times \mathfrak{S}_{\{e_2 - e_3, e_1 - e_4\}}$. If $we_3 = e_3$ and $we_4 = e_4$ then $w(e_1 - e_4) = we_1 - e_4 \in \{e_2 - e_3, e_1 - e_4\}$, which forces $we_1 = e_1$ and similarly $we_2 = e_2$. So $w = \text{id}$ and $\varepsilon(w) = 1$. If w transposes e_3 and e_4, we see in the same way that it also transposes e_1 and e_2, so that $\varepsilon(w) = 1$.

7.5. Asymptotic expansions

Proof of proposition 4.1. Let us deal with the orthogonal case first. We start with the formula $\mathbb{P}_x(T > t) = \sum_{A\in\mathcal{I}} \varepsilon_A \prod_{\alpha\in A} \gamma\left(\hat\alpha(x)/\sqrt{t}\right)$. Expanding γ in power series, noting $l = |I|$ and $a_p = \sqrt{\frac{2}{\pi}}\left(\frac{-1}{2}\right)^p \frac{1}{p!(2p+1)}$, we have

$$\begin{aligned}
\mathbb{P}_x(T > t) &= \sum_{A\in\mathcal{I}} \varepsilon_A \prod_{\alpha\in A} \sum_{p\geq 0} a_p \left(\frac{\hat\alpha(x)}{\sqrt{t}}\right)^{2p+1} \\
&= \sum_{A=\{\alpha_1,\dots,\alpha_l\}\in\mathcal{I}} \varepsilon_A \sum_{p\in\mathbb{N}^l} \prod_{j=1}^{l} a_{p_j} \left(\frac{\hat{\alpha_j}(x))}{\sqrt{t}}\right)^{2p_j+1} \\
&= \sum_{r\geq 0} F_r(x) \left(\frac{1}{\sqrt{t}}\right)^{2r+l},
\end{aligned}$$

where $F_r(x) = \sum_{A\in\mathcal{I}} \varepsilon_A\, G_r(x, A)$ and, if $A = \{\alpha_1, \dots, \alpha_l\}$,

$$G_r(x, A) = \sum_{p\in\mathbb{N}^l,\, \sum_i p_i = r}\ \prod_{j=1}^{l} a_{p_j}\, \hat{\alpha_j}(x)^{2p_j+1}.$$

Note that, unlike the term $\prod_{j=1}^{l} \hat{\alpha_j}(x)^{2p_j+1}$, the previous sum only depends on A and not on the enumeration $\alpha_1, \dots, \alpha_l$ of its elements so the notation $G_r(x, A)$ is legitimate. Now, lemma 7.5 applies to $F_r(x)$ which is a polynomial of degree $2r + l$ so that $F_r(x) = 0$ if $2r + l < n$ and $F_{n+p}(x) = h(x)\, E'_p(x)$ if $p \geq 0$ where $E'_p(x)$ is a W-invariant polynomial of degree p. Since F_r has degree $2r + l$ and $n \equiv l \mod 2$, $E'_p = 0$ if p is odd and we set $E_q = E'_{2q}$ to conclude the proof.

In the semi-orthogonal case, we will call m (resp. l) the number of pairs (resp. singlets) of I^*, so that $|I| = 2m + l$. Whenever we write a pair (α, β), it will be normalized so that it is isometric to $(e_1 - e_2, e_2)$, so that $\mathbb{P}_x(T_{\alpha,\beta} > t) = H\left((x, \alpha+\beta)/\sqrt{t}, (x, \beta)/\sqrt{t}\right)$ with H defined in (4.14). So, writing $(x, \alpha+\beta) = x_{\alpha\beta}$, $(x, \beta) = x_\beta$ and $(x, \gamma) = x_\gamma$,

$$\begin{aligned}
\mathbb{P}_x(T > t) = \sum_{A\in\mathcal{I}} \varepsilon_A &\prod_{(\alpha,\beta)\in A^*} \sum_{p,q\geq 0} a_{pq} \left(\frac{x_{\alpha\beta}x_\beta}{t}\right)^{2p+1} \left(\left(\frac{x_{\alpha\beta}}{\sqrt{t}}\right)^{2q} - \left(\frac{x_\beta}{\sqrt{t}}\right)^{2q}\right) \\
&\prod_{\{\gamma\}\in A^*} \sum_{s\geq 0} a_s \left(\frac{x_\gamma}{\sqrt{t}}\right)^{2s+1} \\
= \sum_{r\geq 0} F_r(x) &\left(\frac{1}{\sqrt{t}}\right)^{r},
\end{aligned}$$

where $F_r(x) = \sum_{A\in\mathcal{I}} \varepsilon_A G_r(x, A)$ and, for $A^* = \{(\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m), \{\gamma_1\}, \dots, \{\gamma_l\}\}$,

$$G_r(x, A) = \sum_{p,q,s} \prod_{j=1}^{m} a_{p_j q_j} (x_{\alpha_j\beta_j}\, x_{\beta_j})^{2p_j+1} \left(x_{\alpha_j\beta_j}^{2q_j} - x_{\beta_j}^{2q_j}\right) \prod_{i=1}^{l} a_{s_i} x_{\gamma_i}^{2s_i+1},$$

where the sum runs over those $p, q \in \mathbb{N}^m$, $s \in \mathbb{N}^l$ such that $4|p| + 2|q| + 2|s| + 2m + l = r$. Again, the previous expression does not depend on the enumeration of pairs (α_j, β_j) or of singlets $\{\gamma_i\}$. Now,

$$(x_{\alpha_j\beta_j}\, x_{\beta_j})^{2p_j+1}(x_{\alpha_j\beta_j}^{2q_j} - x_{\beta_j}^{2q_j})$$

is sign changed when x is replaced by $s_{\alpha_j}x$ or $s_{\beta_j}x$, so that lemma 7.5 applies to $F_r(x)$ which is a polynomial of degree r, null when $r < n$ and divisible by h otherwise. Write $F_{p+n}(x) = h(x)E'_p(x)$ for $p \geq 0$. We remark that $F_r = 0$ if $r \not\equiv l \pmod 2$. Since $n \equiv l \pmod 2$, $E'_p(x) = 0$ if p is odd. Hence the conclusion with $E_q = E'_{2q}$. □

Remark 7.1. *Actually, (4.16) can be directly obtained by expanding each term in (2.2) as a power series in t. As in the previous proof, the factor $h(x)$ appears thanks to the skew-symmetry coming from the alternating sum over W. This approach gives an alternative expression for the polynomials E_q.*

7.6. de Bruijn formulae

Proof of proposition 5.1. Without loss of generality, we can assume that f is continuous and compactly supported, since these functions are dense in $L_1(V)$. Integration of formula (2.1) yields

$$\mathbb{P}_x(T > t) = \int_C \sum_{w\in W} \varepsilon(w) p_t(x, wy)\, dy.$$

For $A \in \mathcal{I}$, we have similarly

$$\mathbb{P}_x(T_A > t) = \int_{C_A} \sum_{w\in W_A} \varepsilon(w) p_t(x, wy)\, dy.$$

Now, using formula (2.3), we get

$$\int_C \sum_{w\in W} \varepsilon(w) p_t(x, wy)\, dy = \sum_{A\in\mathcal{I}} \varepsilon_A \sum_{w\in W_A} \varepsilon(w) \int_{C_A} p_t(x, wy)\, dy.$$

Integrating this with respect to $f(x)\, dx$ and using Fubini's theorem,

$$\int_C \sum_{w\in W} \varepsilon(w) P_t f(wy)\, dy = \sum_{A\in\mathcal{I}} \varepsilon_A \sum_{w\in W_A} \varepsilon(w) \int_{C_A} P_t f(wy)\, dy,$$

where $P_t f(z) = \int f(x) p_t(x, z)\, dx$ is the Brownian semi-group. Now let $t \to 0$ in the above formula and use the fact that $P_t f$ converges in $L_1(V)$ to f to get the result. □

7.6.1. type A

Proof of proposition 5.3. Let us suppose k is even. $\mathcal{I}$ is identified with $P_2(k)$. If $A \in \mathcal{I}$ corresponds with $\pi \in P_2(k)$, then

$$\eta \in \{\pm 1\}^\pi \mapsto w_\eta = \prod_{\{i<j\}\in\pi} \tau_{ij}^{\eta'_{ij}} \in W_A$$

is an isomorphism, where τ_{ij} is the transposition of i and j and $\eta'_{ij} = (1-\eta_{ij})/2$. Then, C_A corresponds with

$$C_\pi = \cap_{\{i<j\}\in\pi}\{y: \ y_i > y_j\}.$$

Since $f(y) = \prod_{\{i<j\}\in\pi} f_i(y_i) f_j(y_j)$, we have

$$f(w_\eta y) = \prod_{\{i<j\}\in\pi,\, \eta_{ij}=1} f_i(y_i) f_j(y_j) \prod_{\{i<j\}\in\pi,\, \eta_{ij}=-1} f_i(y_j) f_j(y_i).$$

Thus, the right-hand side of (5.1) reads

$$\begin{aligned}
&\sum_{\eta\in\{\pm1\}^\pi} \prod_{\{i<j\}\in\pi} \eta_{ij} \int_{C_\pi} \prod_{\{i<j\}\in\pi,\, \eta_{ij}=1} f_i(y_i) f_j(y_j) \prod_{\{i<j\}\in\pi,\, \eta_{ij}=-1} f_i(y_j) f_j(y_i)\, dy \\
&= \sum_{\eta\in\{\pm1\}^\pi} \prod_{\{i<j\}\in\pi,\, \eta_{ij}=1} \int_{y>z} f_i(y) f_j(z)\, dydz \prod_{\{i<j\}\in\pi,\, \eta_{ij}=-1} -\int_{y>z} f_i(z) f_j(y)\, dydz \\
&= \prod_{\{i<j\}\in\pi} \left(\int_{y>z} f_i(y) f_j(z)\, dydz - \int_{y>z} f_i(z) f_j(y)\, dydz \right) = \prod_{\{i<j\}\in\pi} I(f_i, f_j).
\end{aligned}$$

On the other hand hand, the left-hand side of (5.1) is

$$\int_C \sum_{\sigma\in\mathfrak{S}_k} \varepsilon(\sigma) \prod_{i=1}^k f_i(y_{\sigma(i)}) = \int_C \det\left(f_i(y_j)\right)_{i,j\in[k]} dy,$$

which concludes the proof. The case of odd k is treated similarly. □

7.6.2. type D

Proof of proposition 5.4. A first remark is that

$$\int_C \det\left(f_i(y_j)\right)_{i,j\in[k]} dy = 0$$

for odd functions $f_i : \mathbb{R} \to \mathbb{R}$, $1 \le i \le k$. This can be seen by the change of variable $y_k \to -y_k$ which leaves C invariant and changes the sign of the determinant in the previous integral. So we deal with even functions f_i and suppose k is even. $\mathcal{I}$ is identified with $P_2(k)$. If $A \in \mathcal{I}$ corresponds with $\pi \in P_2(k)$, then

$$(\eta, \bar\eta) \in \{\pm1\}^\pi \times \{\pm1\}^\pi \mapsto w_\eta = \prod_{\{i<j\}\in\pi} \tau_{ij}^{\eta'_{ij}} \left(-\tau_{ij}\right)^{\bar\eta'_{ij}} \in W_A$$

is an isomorphism, where τ_{ij} is the transposition of i and j and $\eta'_{ij} = (1-\eta_{ij})/2$, $\bar{\eta_{ij}}' = (1-\bar{\eta_{ij}})/2$. Then, C_A corresponds with

$$C_\pi = \cap_{\{i<j\}\in\pi}\{y:\ y_i > |y_j|\}.$$

Writing $\prod'$ for $\prod_{\{i<j\}\in\pi}$,

$$\begin{aligned} f(w_\eta y) = \prod'_{\eta_{ij}=1,\,\bar{\eta}_{ij}=1} f_i(y_i)f_j(y_j) \prod'_{\eta_{ij}=-1\,\bar{\eta}_{ij}=-1} f_i(-y_i)f_j(-y_j) \\ \prod'_{\eta_{ij}=1,\,\bar{\eta}_{ij}=-1} f_i(-y_j)f_j(-y_i) \prod'_{\eta_{ij}=-1,\,\bar{\eta}_{ij}=1} f_i(y_j)f_j(y_i). \end{aligned}$$

Since the f_i are even, $\sum_{w\in W_A}\varepsilon(w)\int_{C_A} f(wy)\,dy$ can be expressed

$$\begin{aligned} &\sum_{\eta,\,\bar{\eta}\in\{\pm1\}^\pi} \prod' \eta_{ij}\,\bar{\eta}_{ij} \int_{C_\pi} \prod'_{\eta_{ij}=\bar{\eta}_{ij}} f_i(y_i)f_j(y_j) \prod'_{\eta_{ij}=-\bar{\eta}_{ij}} f_i(y_j)f_j(y_i)\,dy \\ &= 2^{|\pi|} \sum_{\hat{\eta}\in\{\pm1\}^\pi} \prod'_{\hat{\eta}_{ij}=1} \int_{y>|z|} f_i(y)f_j(z)\,dydz \prod'_{\hat{\eta}_{ij}=-1} -\int_{y>z} f_i(z)f_j(y)\,dydz \\ &= 2^{|\pi|} \prod_{\{i<j\}\in\pi} K(f_i,f_j). \end{aligned}$$

Hence, the right-hand side of (5.1) is

$$2^{k/2} \sum_{\pi\in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\}\in\pi} K(f_i,f_j).$$

On the other hand hand, the integrand of the left-hand side of (5.1) is

$$\begin{aligned} &\sum_{\sigma\in\mathfrak{S}_k} \varepsilon(\sigma) \sum_{\eta\in\{-1,1\}^k} \frac{1+\prod_i \eta_i}{2} \prod_i f_i(\eta_i y_{\sigma(i)}) \\ &= \frac{1}{2}\left\{\det\left(f_i(y_j)+f_i(-y_j)\right)_{i,j\in[k]} + \det\left(f_i(y_j)-f_i(-y_j)\right)_{i,j\in[k]}\right\} \\ &= 2^{k-1}\det\left(f_i(y_j)\right)_{i,j\in[k]}, \end{aligned}$$

which concludes the proof. The case of odd k is treated similarly. □

7.7. Random walks and related combinatorics

Proof of proposition 6.1. Formula (2.6) reads

$$\mathbb{P}_x(T>t) = \sum_{\pi\in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\}\in\pi} \mathbb{P}_x(X_i - X_j > 0 \text{ on } [0,t]).$$

Under $\mathbb{P}_x$, $X_i - X_j$ is the simple symmetric continuous-time random walk of rate 2 started at $x_i - x_j$. Now recall that, if Z is a simple symmetric continuous-time random walk of rate 2 started at 0, the fixed-time marginals can be computed:

$$\mathbb{P}(Z(t) = x) = p_x(t) \; x \in \mathbb{Z}, \tag{7.6}$$

and there is an analogue of the classical reflection principle:

$$\mathbb{P}(Z \text{ does not reach } x \text{ before time } t) = q_x(t) = \sum_{l=-x+1}^{x} p_l(t), \;\; x \in \mathbb{N}, \tag{7.7}$$

which concludes the proof. □

Proof of proposition 6.2. In (6.1), let us specialize the starting point x to be $\delta = (k-1, k-2, \dots, 1, 0)$. If k is even, each $\pi \in P_2(k)$ has $k/2$ blocks, so that

$$\mathbb{P}_\delta(T > t) = e^{-kt} \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\} \in \pi} \gamma_{j-i}(t) = e^{-kt} \, H_k\left(\gamma(t)\right). \tag{7.8}$$

If k is odd, each $\pi \in P_2(k)$ has $(k-1)/2$ blocks, so that

$$\mathbb{P}_\delta(T > t) = e^{-(k-1)t} \, H_k\left(\gamma(t)\right). \tag{7.9}$$

Now, introduce the "weak" chamber $\Omega_k = \{x : \; x_1 \geq \cdots \geq x_k\}$ and the associated exit time for X, T_{Ω_k}. By translation, we have $\mathbb{P}_\delta(T > t) = \mathbb{P}(T_{\Omega_k} > t)$, where $\mathbb{P}$ governs the process X starting from 0. If we denote by N_t the number of jumps of X before time t, then

$$\mathbb{P}(T_{\Omega_k} > t) = \sum_{n \geq 0} \mathbb{P}(T_{\Omega_k} > t \mid N_t = n) \, \mathbb{P}(N_t = n).$$

Now, $(N_t, t \geq 0)$ is a Poisson process of parameter k, so $\mathbb{P}(N_t = n) = e^{-kt}(kt)^n/n!$, and $\mathbb{P}(T_{\Omega_k} > t \mid N_t = n) = \mathbb{P}(\xi(1), \dots, \xi(n) \in \Omega_k)$, where $(\xi(i), i \in \mathbb{N})$ is a random walk in $\mathbb{N}^k$, starting at 0, with increments uniformly distributed on the canonical basis vectors $\{e_1, \dots, e_k\}$. Hence,

$$\mathbb{P}(T_{\Omega_k} > t \mid N_t = n) = \frac{1}{k^n} \; |\{ \text{ paths } 0 \nearrow \lambda^1 \nearrow \cdots \nearrow \lambda^n \}|,$$

where $\lambda^1, \dots, \lambda^n$ are partitions with at most k parts and $\mu \nearrow \lambda$ means that λ is obtained from μ by adding a box. Seeing the bijection between such sequences of partitions and elements of $\mathcal{T}_k(n)$, we eventually have

$$\mathbb{P}(T_{\Omega_k} > t) = e^{-kt} y_k(t), \tag{7.10}$$

which concludes the proof. □

Proof of proposition 6.3. Follows from proposition 6.2 by de-Poissonization. □

8. Appendix

8.1. A direct proof for A_3

For curiosity, we include a direct proof of formula (4.6) for $k = 4$, involving only probabilistic arguments about path reflections. This proof applies to any strong Markov process X with continuous trajectories and invariant in law under $\mathfrak{S}_k$. Let us first recall how this is done for $k = 3$, in which case the argument comes from [20]. We have $X = (X_1, X_2, X_3) \in \mathbb{R}^3$ and we will refer to X_i as the particle i. Define:

$$T_{ij} = \inf\{t : X_i(t) = X_j(t)\},\ E_{ij} = \{T_{ij} \le t\},\ q_{ij} = \mathbb{P}(E_{ij}),\ T = \inf_{ij} T_{ij}.$$

Our goal is to compute $\mathbb{P}(T \le t)$. Since $E_{13} \subset E_{12} \cup E_{23}$, it follows that $\mathbb{P}(T \le t) = \mathbb{P}(E_{12} \cup E_{23}) = q_{12} + q_{23} - \mathbb{P}(E_{12}\ ,\ E_{23})$. Now, we split according to the first collision and then use the strong Markov property and the invariance in law under any permutation of the particles to switch two particles after their first collision:

$$\begin{aligned}\mathbb{P}(E_{12}\ ,\ E_{23}) &= \mathbb{P}(E_{12}, E_{23}, T = T_{12}) + \mathbb{P}(E_{12}, E_{23}, T = T_{23})\\ &= \mathbb{P}(E_{12}, E_{13}, T = T_{12}) + \mathbb{P}(E_{13}, E_{23}, T = T_{23})\\ &= \mathbb{P}(E_{13}, T = T_{12}) + \mathbb{P}(E_{13}, T = T_{23}) = \mathbb{P}(E_{13}).\end{aligned}$$

Hence, $\mathbb{P}(T \le t) = q_{12} + q_{23} - q_{13}$. This is consistent with (2.10). Then, we easily get $\mathbb{P}(T > t) = p_{12} + p_{23} - p_{13}$.

For $k = 4$, let us keep the same notations and the same reasoning:

$$\begin{aligned}\mathbb{P}(T \le t) &= \mathbb{P}(E_{12} \cup E_{23} \cup E_{34})\\ &= q_{12} + q_{23} + q_{34} - \mathbb{P}(E_{12}, E_{23}) - \mathbb{P}(E_{23}, E_{34}) - \mathbb{P}(E_{12}, E_{34})\\ &\quad + \mathbb{P}(E_{12}, E_{23}, E_{34})\\ &= q_{12} + q_{23} + q_{34} - q_{13} - q_{24} - q_{12}q_{34} + \mathbb{P}(E_{12}, E_{23}, E_{34}),\end{aligned}$$

where we used independence and the previous result for three particles. Now, denoting $E' = E_{12} \cap E_{23} \cap E_{34}$, $\mathbb{P}(E')$ can be split into $\mathbb{P}(E', T = T_{12}) + \mathbb{P}(E', T = T_{23}) + \mathbb{P}(E', T = T_{34})$. Then, switch particles 1 and 2 after T_{12} to get

$$\begin{aligned}\mathbb{P}(E', T = T_{12}) &= \mathbb{P}(E_{12}, E_{13}, E_{34}, T = T_{12})\\ &= \mathbb{P}(E_{13}, E_{34}, T = T_{12})\\ &= \mathbb{P}(E_{14}, T = T_{12}),\end{aligned}$$

where we used the result for $k = 3$ to obtain the last line. Similarly,

$$\mathbb{P}(E', T = T_{34}) = \mathbb{P}(E_{14}, T = T_{34}).$$

Now, the term $\mathbb{P}(E', T = T_{23})$ is dealt with by switching particles 2 and 3 after T_{23},

$$\begin{aligned}\mathbb{P}(E', T = T_{23}) &= \mathbb{P}(E_{13}, E_{23}, E_{24}, T = T_{23})\\ &= \mathbb{P}(E_{13}, E_{24}, T = T_{23})\\ &= \mathbb{P}(E_{13}, E_{24}) - \mathbb{P}(E_{13}, E_{24}, T = T_{12}) - \mathbb{P}(E_{13}, E_{24}, T = T_{34}).\end{aligned}$$

Again, use independence and exchange particles 1, 2 after T_{12} as well as particles 3, 4 after T_{34},

$$\begin{aligned}\mathbb{P}(E', T_{23}) &= q_{13}q_{24} - \mathbb{P}(E_{23}, E_{14}, T = T_{12}) - \mathbb{P}(E_{14}, E_{23}, T = T_{34}) \\ &= q_{13}q_{24} - \mathbb{P}(E_{23}, E_{14}, T \neq T_{23}) \\ &= q_{13}q_{24} - \mathbb{P}(E_{23}, E_{14}) + \mathbb{P}(E_{23}, E_{14}, T = T_{23}) \\ &= q_{13}q_{24} - q_{23}q_{14} + \mathbb{P}(E_{14}, T = T_{23}).\end{aligned}$$

Gathering terms,

$$\mathbb{P}(E_{12}, E_{23}, E_{34}) = q_{14} + q_{13}q_{24} - q_{23}q_{14},$$

which, in turn, yields

$$\mathbb{P}(T \le t) = q_{12} + q_{23} + q_{34} - q_{13} - q_{24} + q_{14} - q_{12}q_{34} + q_{13}q_{24} - q_{14}q_{23}.$$

This is consistent with (2.10) and we easily get

$$\mathbb{P}(T > t) = p_{12}p_{34} + p_{14}p_{23} - p_{13}p_{24}.$$

8.2. The Pfaffian

If $\operatorname{car} \mathbb{K} \neq 2$, any skew-symmetric matrix $A \in \mathcal{M}_n(\mathbb{K})$ can be written $A = PDP^\top$ with $P \in GL(n, \mathbb{K})$, $D = \operatorname{diag}(B_1, \dots, B_q)$ and $B_l = 0 \in \mathbb{K}$ or $B_l = J = (j - i)_{1 \le i, j \le 2} \in \mathcal{M}_2(\mathbb{K})$. Hence, if n is odd, $\det A = 0$. If n is even, one can use the previous decomposition to prove

Proposition 8.1. *There exists a unique polynomial* $\mathrm{Pf} \in \mathbb{Z}[X_{ij}, 1 \le i < j \le n]$ *such that if $A = (a_{ij})$ is a skew-symmetric matrix of size n,* $\det A = \mathrm{Pf}(A)^2$ *and* $\mathrm{Pf}(\operatorname{diag}(J, \dots, J)) = 1$.

The Pfaffian has an explicit expansion in terms of the matrix coefficients:

Proposition 8.2.

$$\mathrm{Pf}(A) = \sum_{\pi \in P_2(n)} (-1)^{c(\pi)} \prod_{\{i<j\} \in \pi} a_{ij} = \frac{1}{2^n (n/2)!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \prod_{i=1}^{n-1} a_{\sigma(i)\sigma(i+1)}.$$

For more on Pfaffians and their properties, see [13, 22].

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