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Concentration results for a Brownian directed percolation problem

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Abstract

We consider the hydrodynamic limit for a certain Brownian directed percolation model, and establish uniform concentration results. In view of recent work on the connection between this directed percolation model and eigenvalues of random matrices, our results can also be interpreted as uniform concentration results at the process level for the largest eigenvalue of Hermitian Brownian motion.

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1. Introduction

Let $B^{(i)}$ denote a sequence of independent Brownian motions in $\mathbb{R}$ and for $t \geqslant 0$ set

$$L_n(t) = \sup_{0=s_0 \leqslant s_1 \leqslant \cdots \leqslant s_{n-1} \leqslant t} \sum_{i=1}^{n} B^{(i)}_{(s_{i-1},s_i)},$$

where $B^{(i)}_{(s,t)} = B^{(i)}_t - B^{(i)}_s$. The random variable $L_n(t)$ can be thought of as a last-passage time for a continuous model of directed percolation. Note that, by Brownian scaling, $L_n(t)/\sqrt{t}$ has the same law as $L_n(1)$, for any $t \geqslant 0$. In Baryshnikov (2001) and Gravner et al. (2001) it is shown that the random variable $L_n(1)$ has the same law as the largest

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eigenvalue of a $n \times n$ GUE random matrix. It therefore follows from standard results in random matrix theory that:

Theorem 1. *For each* $t \geqslant 0$, *as* $n \to \infty$,

$$\frac{1}{n} L_n(nt) \to 2\sqrt{t},$$

in probability.

Our purpose in writing this paper is twofold. Firstly, we will give an alternative proof of this limit theorem using a representation for the process L_n in terms of a sequence of Brownian queues in tandem, and moreover obtain a sharp concentration inequality at the process level. Our second objective is to explore the asymptotic shape of the level sets of $L_n(t)$ using such concentration results.

An outline of our proof of Theorem 1 was given in O'Connell and Yor (2001); it is based on a technique introduced by Seppäläinen (1998) which exploits a kind of convex duality between density and speed in microscopic models for hydrodynamic systems. (For a survey of how this technique can be applied to a range of discrete directed percolation problems, see O'Connell, 1999.) To make the arguments given in O'Connell and Yor (2001) precise, we need a strong uniform (in t) concentration result for the process L_n, and to obtain this we appeal to the general theory of Gaussian processes. Our main result, which we prove in Section 2, is the following refinement of Theorem 1:

Theorem 2. *There exists a sequence* $\{\varepsilon_n\}$ *with* $\varepsilon_n \to 0$ *as* $n \to \infty$ *and constants* $C_1, C_2 > 0$ *such that*

$$\mathbb{P}\left(\sup_{t>0}\left|\frac{(1/n)L_n(nt) - 2\sqrt{t}}{1+t}\right| > y\right) \leqslant C_1 \exp(-C_2 n(y-\varepsilon_n)^2)$$

for all n, y *with* $y > \varepsilon_n$.

At the cost of excluding values of t which are extremely small or extremely large compared to n, one can prove a similar concentration inequality on a finer scale, replacing the denominator $(1+t)$ in Theorem 2 by $\sqrt{t}$:

Theorem 3. *Let* $\gamma < 1$ *let* $g_n = \exp(\exp n^{\gamma})$. *Then there exists a sequence* $\{\varepsilon_n\}$ (*depending on* γ) *with* $\varepsilon_n \to 0$ *as* $n \to 0$, *and constants* $M, C_3, C_4 > 0$ *such that*

$$\mathbb{P}\left(\sup_{g_n^{-1} \leqslant t \leqslant g_n}\left|\frac{(1/n)L_n(nt) - 2\sqrt{t}}{\sqrt{t}}\right| > y\right) \leqslant C_3 \exp(-C_4 n(y-\varepsilon_n)^2)$$

for all n, y *with* $\varepsilon_n \leqslant y \leqslant M$.

In Section 3, we prove this result and use it to analyse a "Brownian growth model" which corresponds to the "corner growth model" associated with discrete last-passage percolation. For the case where the weights in the discrete peroclation problem are exponential (or geometric), the analysis of the limiting shape for this corner growth model dates back at least to Rost (1981); more recently Johansson (2000) derived the exact limiting behaviour of the fluctuations of the shape, using a connection with GUE random matrices, following the analysis in Baik et al. (1999) of the closely related model of the longest increasing subsequence of a random permutation. In Martin (2002), shape theorems are given for models with more general weight distribution, and a universality property for the asymptotics of the limiting shape close to the boundary of the quadrant is proved, using a Brownian scaling related to the one studied here.

To define our growth model, we consider sets defined, for $s \geqslant 0$, by

$$A_s = \{(n,t) \in \mathbb{N} \times \mathbb{R}_+ \colon L_n(t) \leqslant s\}.$$

We can regard s as a time parameter, and describe A_s as the "shape at time s"; this introduces a growth model which corresponds to the Brownian percolation model. One can approximate the random set A_s by the deterministic set H_s, where

$$H_s = \{(n,t) \colon 2\sqrt{nt} \leqslant s\}.$$

We establish shape theorems with uniform convergence. Firstly we show that, for any s, the approximation H_s to A_s becomes arbitrarily accurate as n becomes large, except on a set where t is very small or very large compared to n; this formulation reflects the Brownian scaling inherent in the model. Secondly, we characterise the convergence of a scaled version of A_s to H_1 as s becomes large; this formulation resembles more closely a shape theorem for a traditional discrete first-passage (or last-passage) percolation growth model (e.g. Kesten, 1986).

We remark that the process L_n has the same law as the largest eigenvalue of n-dimensional Hermitian Brownian motion; this was established in O'Connell and Yor (2002). (See also Bougerol and Jeulin, 2001.)

In this setting, out concentration inequalities complement recent large deviations results obtained in Ben Arous and Guionnet (1997), Ben Arous et al. (2001) and Cabanal Duvillard and Guionnet (2001). Let $\{H_n(t),\ t \in \mathbb{R}\}$ be a standard Hermitian Brownian motion, and denote the eigenvalues of $H_n(t)$ by $\lambda_1^n(t) > \cdots > \lambda_n^n(t)$. Then the processes L_n and λ_1^n have the same law, and Theorem 2 is true as stated with L_n replaced by λ_1^n. In the random matrix context, our proof is not very direct; an alternative route would be to prove Theorem 2 using concentration results for Hermitian Brownian motion. However, we have not chosen to pursue this route as we wished to derive the result directly as outlined in O'Connell and Yor (2001).

We mention here briefly some of the concentration and large deviations results which have been obtained in the random matrix context. In Cabanal Duvillard and Guionnet (2001), large deviations upper and lower bounds are obtained for the process empirical measure

$$\sigma_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i^n}$$

at the speed n^2. These upper and lower bounds are sharp when restricted to the marginal at time 1, that is, they agree with the full LDP obtained in Ben Arous and Guionnet (1997), for $\sigma_n(1)$. This yields the correct speed for deviations of λ_1^n *to the left* of its mean. For example, one can compute the limit

$$\lim_{n\to\infty} \frac{1}{n^2} \log P(\lambda_1^n(1) < xn) = -I(x)$$

and $I(x) > 0$ for $x < 2$. For deviations to the right, however, the correct speed is n; a slight modification of the proof of Theorem 6.2 in Ben Arous et al. (2001) yields: for $x > 2$

$$\lim_{n\to\infty} \frac{1}{n} \log P(\lambda_1^n(1) > xn) = -J(x),$$

where $J(x) > 0$. The function J can be computed explicitly. In a recent paper, Ledoux (2002) has obtained very precise concentration results for largest eigenvalue problems.

2. Proof of Theorem 2

In this section we present a proof of Theorem 1 following the outline given in O'Connell and Yor (2001). The idea is to use a representation for the process L_n as a sequence of Brownian queues in tandem, together with the sharp concentration inequality stated in Theorem 2 above.

Lemma 4. *The function $L_n(nt)$ is superadditive and*

$$\frac{1}{n} L_n(nt) \to c\sqrt{t} \text{ a.s.} \quad \text{as } n \to \infty$$

for some constant $c < \infty$.

Proof. Consider an extension of the function of interest

$$L_{n,m}(u,t) = \sup_{u \leqslant s_1 \leqslant \cdots \leqslant s_{m-n-1} \leqslant t} B^{(n+1)}_{(u,s_1)} + \cdots + B^{(m)}_{(s_{m-n-1},t)}.$$

Thus $L_n(nt) := L_{0,n}(0,nt)$. By observation we have the fundamental inequality

$$L_{0,n+m}(0,t) \geqslant L_{0,n}(0,s) + L_{n,n+m}(s,t) \quad \forall s,\ t \in \mathbb{R},\ n,m \in \mathbb{N}. \tag{1}$$

Inequality (1) shows that $L_n(nt)$ is superadditive for fixed t. The stationarity inherent in our set up ensures that the conditions of Kingman's subadditive ergodic theorem are met; hence, there is a function $l(t)$ such that for any t

$$\frac{1}{n} \mathbb{E} L_n(nt) \uparrow l(t) \quad \text{as } n \to \infty$$

and

$$\frac{1}{n} L_n(nt) \to l(t) \text{ a.s.} \quad \text{as } n \to \infty.$$

There is a natural scaling in $L_n(t)$, inherited from the Brownian motion, that

$$L_n(\lambda t) = \sqrt{\lambda} L_n(t) \text{ in distribution.}$$

Thus $\mathbb{E} L_n(nt)/n = c_n\sqrt{t}$, for a sequence of constants with $c_n \uparrow c$ as $n \to \infty$, with the limit $l(t) = c\sqrt{t}$.

Now in O'Connell and Yor (2001) it is shown from a tandem queue representation that for given $m > 0$ one can write

$$\sup_{t>0}\{B_{nt} - mnt + L_{0,n}(-nt,0)\} = \sum_{k=1}^{n} q_k(0), \tag{2}$$

where B is a Brownian motion and $q_k(0)$ are i.i.d. exponential mean $1/m$. Since $L_{0,n}(-nt,0) = L_n(nt)$ in distribution, we can take expectations and divide by n to give

$$\frac{1}{n}\,\mathbb{E}\,L_n(nt) \leqslant \frac{1}{m} + mt$$

for any m and t. But the LHS is $c_n\sqrt{t}$; taking $m = 1/\sqrt{t}$ we get $c_n \leqslant 2$ for all n, and so $c \leqslant 2$ also. $\square$

To show that in fact $c = 2$, we aim to show that the LHS of (2) converges to a Legendre transform of $l(t)$. To do this we establish our concentration inequality. First we need a couple of estimates.

Lemma 5. *For $h > 0$ and any n, t,*

$$\mathbb{P}(|L_n(t) - \mathbb{E}\,L_n(t)| > h) \leqslant 2\exp\left(-\frac{h^2}{2t}\right).$$

Proof. Apply Borell's inequality (e.g. Adler, Theorem 2.1) to the centred Gaussian process $X_{\mathbf{t}} = \sum_{i=1}^{n} B^{(i)}_{t_{i-1},t_i}$ over the parameter set $\{\mathbf{t}\colon \mathbf{t} = (t_1,\ldots,t_{n-1}), 0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_{n-1} \leqslant t_n = t\}$, using the fact that the variance of $X_{\mathbf{t}}$ is t for all $\mathbf{t}$ in the parameter set. $\square$

Lemma 6. *Let $t_0 < t_1$ and $h > 0$. Then for any n,*

$$\mathbb{P}(\textit{For some } t \in (t_0,t_1), L_n(t) \notin (L_n(t_0) - h, L_n(t_1) + h))$$
$$\leqslant 4\,\frac{\sqrt{t_1 - t_0}}{h}\exp\left(-\frac{h^2}{2(t_1 - t_0)}\right).$$

Proof. If $t \in (t_0,t_1)$ then

$$L_n(t_0) + B^{(n)}(t_0,t) \leqslant L_n(t) \leqslant L_n(t_1) - B^{(n)}(t,t_1).$$

So,

$$\mathbb{P}\,(\text{For some } t \in (t_0,t_1), L_n(t) \notin (L_n(t_0) - h, L_n(t_1) + h))$$
$$\leqslant \mathbb{P}(\text{For some } t \in (t_0,t_1), B^{(n)}(t_0,t) < -h \text{ or } B^{(n)}(t,t_1) < -h)$$
$$\leqslant 2\mathbb{P}(M_{t_1-t_0} \geqslant h)$$

(where M_t is the maximum of a Brownian motion over $(0,t)$)

$$= 4\mathbb{P}(B_{t_1-t_0} \geqslant h)$$

by the reflection principle. The result follows from a standard estimate on the tail of the normal distribution. □

Now we prove the concentration inequality in Theorem 2, first for the unknown value of c.

Lemma 7. *There exist constants $C_1, C_2 > 0$ and a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ as $n \to \infty$ such that*

$$P\left(\sup_{t>0}\left|\frac{(1/n)L_n(nt) - c\sqrt{t}}{1+t}\right| > y\right) \leqslant C_1 \exp(-C_2 n(y-\varepsilon_n)^2)$$

for all n, y with $y > \varepsilon_n$.

Proof. Let $c_n = \mathbb{E}L_n(nt)/n\sqrt{t}$ as in the proof of Lemma 4, and define

$$\eta_n(t) = \frac{(1/n)L_n(nt) - c_n\sqrt{t}}{1+t}.$$

Let $\varepsilon_n = (c - c_n) + n^{-1/4}$; note that $\varepsilon_n \downarrow 0$ as $n \to \infty$ since $c_n \uparrow c$. Assume $y \geqslant \varepsilon_n$ and let $x = y - (c - c_n)$. Then

$$\mathbb{P}\left(\sup_{t>0}\left|\frac{(1/n)L_n(nt) - c\sqrt{t}}{1+t}\right| > y\right) \leqslant \mathbb{P}\left(\sup_{t>0}|\eta_n(t)| > x\right). \tag{3}$$

Let $\delta = \min\{1, (x/6c)^2\}$. From the definitions of y, x, ε_n and δ we have that $x > n^{-1/4}$, and that $n\delta x^2 \geqslant (1+6c)^{-2}$. To estimate the RHS of (3), we begin by dividing the t-axis into intervals of length δ. We have

$$\mathbb{P}\left(\sup_{t>0}|\eta_n(t)| > x\right) \leqslant \mathbb{P}\left(\sup_{j\in\mathbb{N}}|\eta_n(j\delta)| > \frac{x}{3}\right) + \mathbb{P}\left(\sup_{t>0}|\eta_n(t)| > x, \sup_{j\in\mathbb{N}}|\eta_n(j\delta)| \leqslant \frac{x}{3}\right).$$

We estimate these two terms separately.

For the first term, Lemma 5 gives

$$\begin{aligned}\mathbb{P}\left(\sup_{j\in\mathbb{N}}|\eta_n(j\delta)| > \frac{x}{3}\right) &\leqslant \sum_{j\in\mathbb{N}}\mathbb{P}\left(|\eta_n(j\delta)| > \frac{x}{3}\right)\\ &= \sum_{j\in\mathbb{N}}\mathbb{P}\left(|L_n(nj\delta) - \mathbb{E}L_n(nj\delta)| > n(1+j\delta)\frac{x}{3}\right)\\ &\leqslant \sum_{j\in\mathbb{N}} 2\exp\left(-\frac{n(1+j\delta)^2x^2}{18j\delta}\right)\\ &\leqslant 2\sum_{j\in\mathbb{N}}\exp\left(-\frac{n}{18}(1+j\delta)x^2\right).\end{aligned} \tag{4}$$

For the second term, note that

$$\mathbb{P}\left(\sup_{t>0}|\eta_n(t)| > x, \sup_{j\in\mathbb{N}}|\eta_n(j\delta)| \leqslant \frac{x}{3}\right)$$

$$\leqslant \sum_{j\in\mathbb{N}} \mathbb{P}\Bigg(\exists t\in(j\delta,(j+1)\delta):$$

$$\eta_n(t) \notin \left[-2|\eta_n(j\delta)| - \frac{x}{3}, 2|\eta_n((j+1)\delta)| + \frac{x}{3}\right]\Bigg). \tag{5}$$

But a direct calculation from the definitions of η_n and of δ gives the following property: if

$$j\delta < t < (j+1)\delta$$

and

$$L_n(nt)\in\left[L_n(nj\delta) - n(1+j\delta)\frac{x}{6}, L_n(n(j+1)\delta) + n(1+j\delta)\frac{x}{6}\right]$$

then

$$\eta_n(t)\in\left[-2|\eta_n(j\delta)| - \frac{x}{3}, 2|\eta_n((j+1)\delta)| + \frac{x}{3}\right].$$

Thus, using (5) and then Lemma 6,

$$\mathbb{P}\left(\sup_{t>0}|\eta_n(t)| > x, \sup_{j\in\mathbb{N}}|\eta_n(j\delta)| \leqslant \frac{x}{3}\right)$$

$$\leqslant \sum_{j\in\mathbb{N}} \mathbb{P}\Bigg(\exists t\in(j\delta,(j+1)\delta):$$

$$L_n(nt) \notin \left[L_n(nj\delta) - n(1+j\delta)\frac{x}{6}, L_n(n(j+1)\delta) + n(1+j\delta)\frac{x}{6}\right]\Bigg) \tag{6}$$

$$\leqslant \sum_{j\in\mathbb{N}} \frac{4\sqrt{\delta}}{\sqrt{n}(1+j\delta)x/6}\exp\left(-\frac{n(1+j\delta)^2x^2}{72\delta}\right)$$

$$\leqslant 4(1+c)\sum_{j\in\mathbb{N}}\exp\left(-\frac{n}{72}(1+j\delta)x^2\right). \tag{7}$$

Adding the RHS of (4) and (7), we get (for constants C, C_1, C_2)

$$\mathbb{P}\left(\sup_{t>0}|\eta_n(t)| > x\right) \leqslant C\sum_{j\in\mathbb{N}}\exp(-C_2n(1+j\delta)x^2)$$

$$= \frac{C\exp(-C_2nx^2)}{1-\exp(-C_2n\delta x^2)}$$

$$\leqslant C_1\exp(-C_2nx^2) \quad (\text{since } n\delta x^2 \geqslant (1+6c)^{-2})$$

$$\leqslant C_1\exp(-C_2n(y-\varepsilon_n)^2)$$

since $0 \leqslant y - \varepsilon_n \leqslant x$. By (3), we are finished. □

We have now deduced the technical conditions required to prove directly the result of O'Connell and Yor (2001), Section 4.

To complete the proof we introduce a little more notation. Let $\mathscr{C}$ denote the continuous functions on $[0,\infty)$ and let $\mathscr{C}_0 = \mathscr{C} \cap \{\phi\colon\ \phi(0)=0, \lim_{t\to\infty}\phi(t)/t = 0\}$ equipped with the norm $\|\phi\| = \sup_{t\geqslant 0}|\phi(t)|/(1+t)$.

Theorem 8. *For any $t > 0$*

$$L_n(nt)/n \to 2\sqrt{t} \text{ a.s.} \quad \text{as } n\to\infty.$$

Proof. As at (2) we have that

$$\sup_{t>0}\left\{\frac{1}{n}B_{(0,nt)} - mt + \frac{1}{n}L_n(nt)\right\} \to \frac{1}{m} \tag{8}$$

in probability, as $n\to\infty$. We now justify taking the limit on the LHS to obtain a Legendre transform of $l(t)$, where $l(t) = c\sqrt{t}$ is the limit established in Lemma 4. As the map $\phi \to \sup_{t\geqslant 0}\{\phi(t) - \mu t\}$ is continuous on $\mathscr{C}_0$ in the induced topology, by Ganesh and O'Connell (2002), if we can prove

$$\left\|\frac{1}{n}B_{(0,n\cdot)} + \frac{1}{n}L_n(n\cdot) - l\right\| \to 0, \qquad \text{a.s.} \tag{9}$$

as $n\to\infty$, then

$$\sup_{t>0}\left\{\frac{1}{n}B_{(-t,0)} - mt + \frac{1}{n}L_n(t)\right\} \to \sup_{t>0}\{-mt + l(t)\} \tag{10}$$

a.s. as $n\to\infty$. The proof is then completed by comparing (8) and (10) and inverting the Legendre transform.

We write $\tilde{B}_n(t) = B_{(0,nt)}/n$ and $\tilde{L}_n(t) = L_n(nt)/n$ and bound the left side of (9) by two terms as

$$\left\|\frac{1}{n}B_{(0,n\cdot)} + \frac{1}{n}L_n(n\cdot) - l\right\| \leqslant \|\tilde{B}_n\| + \|\tilde{L}_n - l\|.$$

Firstly $\|\tilde{B}_n\| \to 0$ almost surely. We write $M_t = \sup_{0\leqslant s\leqslant t} B_{(0,s)}$ and apply standard facts about Brownian motion, to get

$$\begin{aligned}
P\left(\sup_{t\geqslant 0}\frac{|B_{(0,nt)}|}{n(1+t)} \geqslant x\right) &\leqslant P\left(\sup_{0<t<1}\frac{|B_{(0,nt)}|}{n(1+t)} \geqslant x\right) + P\left(\sup_{1\leqslant t<\infty}\frac{|B_{(0,nt)}|}{n(1+t)} \geqslant x\right)\\
&\leqslant P\left(\frac{1}{n}M_n > x\right) + P\left(\sup_{1\leqslant t<\infty}\frac{nt|B_{(0,1/nt)}|}{n(1+t)} \geqslant x\right)\\
&\leqslant P(M_n > nx) + P(M_{1/n} > 2x)\\
&\leqslant a\exp(-x^2 n),
\end{aligned}$$

for some constant a. This exponential tail estimate gives the almost sure convergence to 0.

For the second term we apply the concentration inequality in Lemma 7. The exponential rate of convergence ensures that we have the almost sure convergence in the norm.

Thus, both terms converge almost surely and we have established the required continuity on $\mathscr{C}_0$ to deduce the convergence of the Legendre transforms. The final part is to invert the Legendre transform, which we leave for the reader. □.

3. Brownian growth model

3.1. Results

For $s \geqslant 0$, define the set

$$A_s = \{(n,t):\ L_n(t) \leqslant s\}$$

considered as a subset of $\mathbb{N} \times \mathbb{R}_+$. We can regard s as a time parameter, and describe A_s as the "shape at item s"; this introduces a growth model which corresponds to the Brownian peroclation model.

By definition of A_s and of $L_n(t)$, we have that

(i) $A_s \subset A_{s'}$ whenever $s < s'$ and
(ii) for any s, $(n',t) \in A_s$ whenever $(n,t) \in A_s$ and $n' < n$.

However, it is not the case that $(n,t') \in A_s$ whenever $(n,t) \in A_s$ and $t' < t$.

We can approximate the random set A_s by the deterministic set H_s, where

$$H_s = \{(n,t):\ 2\sqrt{nt} \leqslant s\}.$$

The first result below, which we prove using Theorem 3, shows that as n becomes large, this approximation becomes arbitrarily accurate (uniformly in s), except on a set of points where t is extremely small or extremely large compared to n:

Theorem 9. *Let $\gamma < 1$, and let $g_n = \exp(\exp(n^\gamma))$. Then w.p. 1*

$$\lim_{n\to\infty} \sup_{\substack{s,t: g_n^{-1} \leqslant t \leqslant g_n \\ 2\sqrt{nt} \leqslant s}} \left\{ \frac{L_n(t)}{s} - 1 \right\} = 0 \tag{11}$$

and

$$\lim_{n\to\infty} \inf_{\substack{s,t: g_n^{-1} \leqslant t \leqslant g_n \\ s \leqslant 2\sqrt{nt}}} \left\{ \frac{L_n(t)}{s} - 1 \right\} = 0. \tag{12}$$

Thus for any $\varepsilon > 0$, w.p. 1 there exists N large enough such that

$$(H_{(1-\varepsilon)s} \cap G_N) \subset (A_s \cap G_N) \subset H_{(1+\varepsilon)s},$$

where G_N is the set $\{(n,t):\ n \geqslant N, g_n^{-1} \leqslant t \leqslant g_n\}$.

If we do not require uniformity in s, we can remove the lower bound in (11):

Theorem 10. *Let $s > 0$. Then w.p. 1*

$$\sup_{n,t:2\sqrt{nt}\leqslant s} L_n(t) < \infty \tag{13}$$

and

$$\lim_{n\to\infty} \sup_{t:2\sqrt{nt}\leqslant s} L_n(t) = s. \tag{14}$$

However, we cannot similarly remove the upper bound in (12). The process $L_n(t)$ can behave badly very close to the t-axis; for example it is the case that

$$\liminf_{t\to\infty} L_n(t) = -\infty \text{ a.s} \quad \text{for all } n.$$

Finally, we give a result which more closely resembles a shape theorem for a more traditional first-passage percolation growth model (e.g. Kesten, 1986, Theorem 1.7):

Theorem 11. *Let $\gamma < 1$ and define $G = \{(n,t)\colon t \leqslant \exp(\exp(n^\gamma))\}$. Then for any $\varepsilon > 0$, w.p. 1,*

$$(H_{(1-\varepsilon)s} \cap G) \subset (A_s \cap G) \subset (H_{(1+\varepsilon)s} \cap G)$$

for all sufficiently large s.

Such a formulation, indicating that the difference between A_s and H_s becomes arbitrarily small as s becomes large, (so equivalently as the product nt becomes large), is less natural in the Brownian context, however. Here the distribution of A_s is the same for all s, up to a linear rescaling; by Brownian scaling, the form of the fluctuations of $L_n(t)$ depend essentially only on n and not on t.

In the remainder of this section we prove Theorem 3 and the shape results above.

3.2. Proof of Theorem 3

Proof. As before, write $c_n = \mathbb{E}L_n(nt)/n\sqrt{t}$, so that $c_n \uparrow 2$.

Let $x = \frac{1}{4}[y - (2 - c_n)]$; then we can bound the quantity we are interested in by

$$\mathbb{P}\left(\sup_{g_n^{-1}\leqslant t\leqslant g_n}\left|\frac{(1/n)L_n(nt) - 2\sqrt{t}}{\sqrt{t}}\right| > y\right)$$

$$\leqslant \mathbb{P}\left(\sup_{g_n^{-1}\leqslant t\leqslant g_n}\left|\frac{L_n(nt) - \mathbb{E}L_n(nt)}{\sqrt{t}}\right| > 4xn\right). \tag{15}$$

Assume for the moment that

$$n^{-(1-\gamma)/4} \leqslant x \leqslant 1 \tag{16}$$

(we will strengthen this assumption later). For $r \in \mathbb{Z}$, denote by I_r the interval

$$[(1+x)^{2(r-1)}, (1+x)^{2r}].$$

Define $R(n) = \exp(n^{(1+\gamma)/2})$. Then using (16) one can show that $(1+x)^{2R(n)} \geqslant g_n$ for all $n \geqslant 1$; thus, the collection of intervals $\{I_r, -R(n) \leqslant r \leqslant R(n)\}$ covers $[g_n^{-1}, g_n]$ for all n.

Fix n and r. To bound the RHS of (15), we will estimate the probability that

$$\left|L_n(nt) - c_n n\sqrt{t}\right| \geqslant 4xn\sqrt{t}$$

for some $t \in I_r$ (and then sum over r). Let $t_0 = (1+x)^{2(r-1)}$ and $t_1 = (1+x)^2 t_0$, so that $I_r = [t_0, t_1]$. Then for all $t, t' \in I_r$, we have that

$$\left|\sqrt{t'} - \sqrt{t}\right| \leqslant x\sqrt{t_0}$$

and so, since $\mathbb{E}\, L_n(nt) = c_n n\sqrt{t} \leqslant 2n\sqrt{t}$,

$$|\mathbb{E}\, L_n(nt) - \mathbb{E}\, L_n(nt')| \leqslant 2xn\sqrt{t_0}. \tag{17}$$

Using Lemma 5, we have that

$$\mathbb{P}(|L_n(nt_0) - \mathbb{E}\, L_n(nt_0)| \geqslant xn\sqrt{t_0}) \leqslant 2\exp\left(-\frac{x^2 n}{2}\right) \tag{18}$$

and that

$$\begin{aligned}\mathbb{P}(|L_n(nt_1) - \mathbb{E}\, L_n(nt_1)| \geqslant xn\sqrt{t_0}) &\leqslant 2\exp\left(-\frac{x^2 n t_0}{2t_1}\right)\\ &= 2\exp\left(-\frac{x^2 n}{2(1+x)^2}\right).\end{aligned} \tag{19}$$

From Lemma 6, we also have that

$$\begin{aligned}&\mathbb{P}(\exists t \in I_r \text{ with } L_n(nt) \notin [L_n(nt_0) - xn\sqrt{t_0},\ L_n(nt_1) + xn\sqrt{t_1}])\\ &\quad \leqslant \frac{4\sqrt{x^2+2x}}{x\sqrt{n}} \exp\left(-\frac{x^2 n}{2(x^2+2x)}\right).\end{aligned} \tag{20}$$

Now if none of the events on the LHS of (18), (19) or (20) occurs, then, using (17)

$$|L_n(nt) - \mathbb{E}\, L_n(nt)| \leqslant 4xn\sqrt{t_0} \leqslant 4xn\sqrt{t}$$

for all $t \in I_r$. So, summing the RHS of (18), (19) and (20) and using the assumption that $x \leqslant 1$, we have

$$\mathbb{P}(\exists t \in I_r \text{ with } |L_n(nt) - \mathbb{E}\, L_n(nt)| > 4xn\sqrt{t}) \leqslant C' \exp(-Cnx^2)$$

for some constants $C, C' > 0$ (independent of x, n, r and γ).

Summing this over all r with $-R(n) < r \leqslant R(n)$, we obtain

$$\begin{aligned}\mathbb{P}\left(\sup_{g_n^{-1} \leqslant t \leqslant g_n} \left|\frac{L_n(nt) - \mathbb{E}\, L_n(nt)}{\sqrt{t}}\right| > 4xn\right) &\leqslant 2R(n)C' \exp(-Cnx^2)\\ &= C'' \exp(-Cnx^2 + n^{(1+\gamma)/2}).\end{aligned} \tag{21}$$

We are essentially done now, except that we need to remove the positive term inside the exponential on the RHS of (21).

To do this, we strengthen the assumption in (16), and now suppose that

$$(1 + C^{-1/2})n^{-(1-\gamma)/4} \leqslant x \leqslant 1.$$

From the definition of x, this is implied by

$$\varepsilon_n \leqslant y \leqslant 4, \tag{22}$$

where

$$\varepsilon_n = (2 - c_n) + 4(1 + C^{-1/2})n^{-(1-\gamma)/4}. \tag{23}$$

(Since $c_n \uparrow 2$, we have $\varepsilon_n \to 0$ as desired.) From (22), (23) and the definition of x, one can proceed to show that

$$-Cnx^2 + n^{(1+\gamma)/2} \leqslant -\frac{C}{16}(y - \varepsilon_n)^2.$$

Plugging this into the RHS to (21) and using (15), we have proved the theorem with $\{\varepsilon_n\}$ defined by (23) and with $M = 4$. □

3.3. *Proof of results from Section 3.1*

Proof of Theorem 9. Using Borel–Cantelli and Theorem 3 for some value γ' with $\gamma < \gamma' < 1$, we have

$$\lim_{n\to\infty} \sup_{\tilde{t}: h_n^{-1} \leqslant \tilde{t} \leqslant h_n} \left| \frac{(1/n)L_n(n\tilde{t}) - 2\sqrt{\tilde{t}}}{\sqrt{\tilde{t}}} \right| = 0 \quad \text{a.s.,}$$

where $h_n = \exp(\exp n^{\gamma'})$. Writing $t = n\tilde{t}$ and using the fact that $ng_n < h_n$ when n is large enough, we can rewrite this to get

$$\lim_{n\to\infty} \sup_{t: g_n^{-1} \leqslant t \leqslant g_n} \left| \frac{L_n(t)}{2\sqrt{nt}} - 1 \right| = 0 \quad \text{a.s.} \tag{24}$$

Now (11) and (12) follow immediately.

To deduce the last part of the theorem, note that if $(n,t) \in H_{(1-\varepsilon)s}$ but $(n,t) \notin A_s$, then $2\sqrt{nt} \leqslant (1-\varepsilon)s$ and $L_n(t) > s$, giving $L_n(t)/2\sqrt{nt} > 1/(1-\varepsilon)$. Similarly, if $(n,t) \in A_s \setminus H_{(1+\varepsilon)s}$, then $L_n(t)/2\sqrt{nt} < 1/(1+\varepsilon)$. In either case there are a.s. only finitely many n such that this occurs for some $t \in [g_n^{-1}, g_n]$. □.

Proof of Theorem 10. Let $s > 0$ and $h > 0$. For any n,

$$\begin{aligned} \mathbb{P}\left(L_n\left(\frac{s^2}{4n}\right) > s(1+h)\right) &\leqslant \mathbb{P}\left(L_n\left(\frac{s^2}{4n}\right) - \mathbb{E}L_n\left(\frac{s^2}{4n}\right) > hs\right) \\ &\leqslant \exp(-2h^2 n), \end{aligned}$$

using Lemma 5 and the fact that $\mathbb{E}L_n(S^2/4n) \leqslant l(s^2/4) = s$.

Also

$$\mathbb{P}\left(\sup_{t:2\sqrt{nt}\leqslant s} L_n(t) > L_n\left(\frac{s^2}{4n}\right) + hs\right) = \mathbb{P}\left(\sup_{0\leqslant t\leqslant \frac{s^2}{4n}} L_n(t) > L_n\left(\frac{s^2}{4n}\right) + hs\right)$$

$$\leqslant \frac{1}{2h\sqrt{n}}\exp(-2h^2 n),$$

by Lemma 6. Hence for any n,

$$\mathbb{P}\left(\sup_{t:2\sqrt{nt}\leqslant s} L_n(t) > s(1+2h)\right) \leqslant \left(1+\frac{1}{2h\sqrt{n}}\right)\exp(-2h^2 n).$$

Now (13) follows since the RHS tends to 0 as $h\to\infty$, while (14) follows from Borel–Cantelli since the sum of the RHS over n is finite for any $h>0$. □

Proof of Theorem 11. If $(n,t)\in H_{(1-\varepsilon)s}$ but $(n,t)\notin A_s$, then $L_n(t)>s$ and $L_n(t)/2\sqrt{nt} > 1/(1-\varepsilon)$.

On the other hand, if $(n,t)\in A_s$ but $(n,t)\notin H_{(1+\varepsilon)s}$, then $2\sqrt{nt}>s$ and $L_n(t)/2\sqrt{nt} < 1/(1+\varepsilon)$.

So to prove the theorem it will suffice to show that for any $\delta>0$, w.p. 1, the set

$$\left\{(n,t)\colon \left|\frac{L_n(t)}{2\sqrt{nt}}-1\right| > \delta; \max\{L_n(t), 2\sqrt{nt}\} > s; t\leqslant \exp(\exp n^\gamma)\right\}$$

is empty for large enough s.

Suppose this fails. Then there is a sequence (n_k,t_k) such that, for each k,

$$\left|\frac{L_{n_k}(t_k)}{2\sqrt{n_k t_k}}-1\right| > \delta$$

and

$$t_k \leqslant \exp(\exp n_k^\gamma) \tag{25}$$

and either $L_{n_k}(t_k)\to\infty$ or $2\sqrt{n_k t_k}\to\infty$.

But by (13), if $L_{n_k}(t_k)\to\infty$ then necessarily (w.p. 1) $2\sqrt{n_k t_k}\to\infty$ anyway. Thus $t_k > n_k^{-1}$ for large enough k, and by (25) if also follows that $n_k\to\infty$. So finally it suffices to show that the set

$$\left\{n\colon \exists t \text{ with } n^{-1}\leqslant t\leqslant \exp(\exp n^\gamma) \text{ and } \left|\frac{L_n(t)}{2\sqrt{nt}}-1\right| > \delta\right\}$$

is a.s. finite. This follows immediately from (24). □

References

Adler, R.J. An Introduction to Continuity Extrema, and Related Topics for General Gaussian Processes, IMS Lecture Notes Monograph Series, Vol. 12. Inst. of Math. Statistics, Hayward, CA, 1990.

Baik, J., Deift, P.A., Johansson, K., 1999. On the distribution of the length of the longest increasing subsequence of random permutation. J. Amer. Math. Soc. 12, 1119–1178.

Baryshnikov, Yu., 2001. GUEs and queues. Probab. Theory Related Fields 119, 256–274.
Ben Arous, G., Guionnet, A., 1997. Large deviations for Wigner's law and Voiculescu's non-commutative entropy. Probab. Theory Related Fields 108, 517–542.
Ben Arous, G., Dembo, A., Guionnet, A., 2001. Aging of spherical spin glasses. Probab. Theory Related Fields 120, 1–67.
Bougerol, Ph., Jeulin, Th., 2001. Paths in Weyl chambers and random matrices. Preprint.
Cabanal Duvillard, T., Guionnet, A., 2001. Large deviations upper bounds for the laws of matrix-valued processes and non-commutative entropies. Ann. Probab. 29, 1205–1261.
Ganesh, A., O'Connell, N., 2002. A large deviation principle with queueing applications. Stochastics Stochastic Rep. 73, 25–35.
Gravner, J., Tracy, C.A., Widom, H., 2001. Limit theorems for height fluctuations in a class of discrete space and time growth models. J. Statist. Phys. 102 (5–6), 1085–1132.
Johansson, K., 2000. Shape fluctuations and random matrices. Commun. Math. Phys. 209, 437–476.
Kesten, H., 1986. Aspects of first passage percolation. In: École d'été de Probabilités de Saint-Flour, XIV-1984, Springer, Berlin, pp. 125–264.
Ledoux, M., 2002. A remark on hypercontractivity and tail inequalities for the largest eigenvalues of random matrices. Preprint.
Martin, J.B., 2002. Limiting shape for directed percolation models. Preprint.
O'Connell, N., 1999. Directed percolation and tandem queues. DIAS Technical Report DIAS-APG-9912.
O'Connell, N., Yor, M., 2001. Brownian analogues of Burke's theorem. Stochastic Process. Appl. 96, 285–304.
O'Connell, N., Yor, M., 2002. A representation for non-colliding random walks. Elec. Commun. Probab. 7, 1–12.
Rost, H., 1981. Nonequilibrium behaviour of a many particle process: density profile and local equilibria. Z. Wahrsch. Verw. Gebiete 58, 41–53.
Seppäläinen, T., 1998. Hydrodynamic scaling, convex duality, and asymptotic shapes of growth models. Markov Process. Related Fields 4, 1–26.