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Weyl Chambers, Symmetric Spaces and Number Variance Saturation

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Abstract. We survey some probabilistic aspects of reflection groups, Weyl chambers, symmetric spaces and the Riemann zeta function, and make new observations on the inter-relationships between these topics. We first consider Brownian motion in the fundamental chamber of a finite reflection group and remark on an interesting feature of this process, namely, that as far as the one-dimensional marginal distributions are concerned, the drift and initial position are interchangeable. We also observe that, when the finite reflection group is a Weyl group, Brownian motion with a particular choice of drift, conditioned to remain in the corresponding Weyl chamber $\mathfrak{a}^+$, has the same distribution as the radial part of Brownian motion on an associated symmetric space. In the type A case, these observations enable us to establish algebraic as well as new probabilistic interpretations for a combinatorial model recently put forward by Johansson (2004) as a means of constructing a point process that emulates the number variance saturation behaviour of the Riemann zeta zeroes. In order to make this account accessible to a wide audience, where appropriate, we discuss known results and outline the relevant background material.

Introduction

It is well established that Brownian motion in a Weyl chamber started at the origin may be interpreted as the radial part of standard Brownian motion on a "flat" symmetric space of Euclidean type (Bougerol and Jeulin (2002); Grabiner (1999)). The purpose of this paper is to highlight the existence of an analogous

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relationship between Brownian motion with drift in a Weyl chamber and the radial part of Brownian motion on a non-compact symmetric space of negative curvature. Our objective is then to use this relationship to give an alternative description of a random matrix model recently employed by Johansson to construct a process which demonstrates number variance saturation behaviour (Johansson (2004)). We present a survey of related material along the way.

Various descriptions of n-dimensional Brownian motion conditioned for its components to never collide and more generally of Brownian motion in a fundamental (Weyl) chamber, have been studied in recent times, see for example Grabiner (1999); O'Connell (2003b); Biane et al. (2005) and references therein. Some such processes have been successfully constructed using purely probablistic concepts and combinatorial methods e.g. Mehta (2004); O'Connell and Yor (2002). However, recent work of Bougerol and Jeulin (2002); Biane et al. (2005) has focused on developing the more general algebraic methods for constructing these processes. In these papers, a path-transformation is introduced which has a representation-theoretic interpretation in the context of complex semisimple Lie algebras and which, when applied to an appropriate Euclidean Brownian motion, yields Brownian motion in a Weyl chamber started at the origin.

In a seemingly separate area of random matrix theory, systems of one-dimensional, non-colliding Brownian motions started from equidistant points on the real line have been considered as models for the spectral fluctuations of quantum systems of mixed type (Forrester (1996); Guhr and Papenbrock (1999)). Johansson (2004) was recently able to exploit such a construction to create a point process with determinantal structure, possessing the additional property that the variance of the number of points in an interval of length L converges to a limiting value as $L \to \infty$. This is a property shared by the Riemann zeta zeroes but not exhibited by previous random matrix models for the zeroes. These constructions, have, until now, been described in combinatorial terms.

Here we explain connections between the aforementioned symmetric space interpretations, path transformations and random matrices. In doing so, we are able to deduce a Gaussian random matrix interpretation for the eigenvalues of Brownian motion on the space of positive-definite Hermitian matrices. This provides a way of viewing Johansson's model from an algebraic perspective and also addresses a question recently put forth in Katori and Tanemura (2004) concerning the existence of such an interpretation for the closely related Brownian motion of ellipsoids (Norris et al. (1986)).

The paper is organised as follows. In the first section, we recall known facts relating to standard Brownian motion in the fundamental chamber of a finite reflection group. We give a brief introduction to the relevant reflection group concepts before presenting a description of this process in terms of Doob's conditioning. We illustrate the description with three "classical" examples.

In the second section we describe how Brownian motion with drift may be conditioned to remain in a fundamental chamber by adjusting the approach described in the first section. Proposition 1, states that interchangeing the drift and initial position has no effect on the one-dimensional marginal distributions of this process. This observation allows us to relate the algebraic descriptions of the conditioned process started at the origin, to the corresponding process started from differing points along the real line, thus giving an alternative interpretation of Johansson's number variance saturation model.

In the third section we consider a special instance of Brownian motion with a very specific drift, conditioned to remain in a Weyl chamber, and describe how this is associated with the radial part of Brownian motion on the corresponding non-compact symmetric space. Proposition 2 makes this connection precise with reference to the general framework of non-compact symmetric spaces associated with complex semisimple Lie algebras. We proceed by carefully examining the implications of this result for the type A case.

In the fourth and final section we review the background and motivation behind Johansson's model and outline some of the results of Johansson (2004). We conclude by clarifying how all the different elements considered in the paper are connected.

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1. Brownian motion in the fundamental chamber of a finite reflection group

For the reader's convenience we start with an outline of definitions and known facts relating to Brownian motion in the fundamental chamber of a finite reflection group.

1.1. *Background on reflection groups.* We begin by setting the scene and explain what is meant by "the fundamental chamber of a finite reflection group". We shall be working with a finite dimensional Euclidean space V with inner product denoted $\langle \cdot, \cdot \rangle$. Our aim in this section will be to use the geometric structures associated with reflections in V to give elegant descriptions of Brownian motion conditioned to behave in particular ways.

Formally, a reflection is a linear operator

$$\begin{aligned} s_\alpha &: V \to V \\ s_\alpha x &:= x - \frac{2\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha \end{aligned} \tag{1.1}$$

It is readily seen that this definition agrees with our everyday notion of reflection in that s_α transforms $\alpha \in V$, $\alpha \neq 0$ into a negative copy of itself but leaves all points of the reflecting hyperplane $H_\alpha := \{ x : \langle x, \alpha \rangle = 0 \}$, orthogonal to α, unchanged.

A finite reflection group, denoted W, is, as the name might suggest, a finite group generated by reflections. Note that since the reflection operator s_α preserves the inner product, reflection groups acting on V can be thought of as specific subgroups of $O(V) := \{ f : V \to V \text{ linear and } \langle f(\alpha), f(\nu) \rangle = \langle \alpha, \nu \rangle \}$, the group of all orthogonal transformations of V. In particular, the inner product on V is invariant under the action of the group W, that is, $\langle w(\alpha), w(\nu) \rangle = \langle \alpha, \nu \rangle$ for all $w \in W$.

The structures of different reflection groups are conveniently described by certain finite sets of non-zero vectors called root systems. In the reflection group context elements of a root system Φ are vectors in V that satisfy the following two axioms

R1 If $\alpha \in \Phi$, then $\lambda\alpha \in \Phi$ iff $\lambda = \pm 1$.

R2 If $\alpha,\, \nu \in \Phi$ then $s_\alpha \nu \in \Phi$

Any such root system can be used to construct a finite reflection group $W(\Phi)$ defined as the group generated by the reflections $\{s_\alpha : \alpha \in \Phi\}$. $W(\Phi)$ is determined completely by its associated root system though different root systems may give rise to the same reflection group.

It turns out that the root system Φ is superfluous to our needs. A "simpler" subset of Φ will be sufficient to reconstruct Φ and hence $W(\Phi)$ when required. A subset $\Sigma \subset \Phi$ serves this purpose and is called a fundamental (simple) system if

(1) Σ is linearly independent.
(2) Every element of Φ can be written as a linear combination of the elements of Σ with coefficients all of the same sign.

The fundamental (Weyl) chambers of $W(\Phi)$ are simply the geometric configurations determined by the way the hyperplanes of reflection associated with the root system are arranged in V. More precisely they are the connected components of $V \setminus (\bigcup_{\alpha\in\Phi} H_\alpha)$. It emerges that there is a one-to-one correspondence between the chambers of $W(\Phi)$ and the fundamental systems of Φ. The fundamental chamber corresponding to a given fundamental system Σ is defined as

$$C_\Sigma := \{x \in V : \langle x, \alpha \rangle > 0 \;\; \forall\, \alpha \in \Sigma\} \tag{1.2}$$

This is the component with "walls" $\{H_\alpha : \alpha \in \Sigma\}$. It is easy to see from the bilinearity of the inner product that C_Σ is a convex cone in V. Replacing Σ by $w\Sigma$, $w \in W(\Phi)$ changes C_Σ to wC_Σ - the chamber is a fundamental domain for the action of $W(\Phi)$ on V.

Each fundamental chamber (or Σ) determines an unique partition of Φ with roots categorised as positive or negative depending on which side of the chamber they lie, i.e.

$$\Phi^+ := \{\nu \in \Phi : \langle \nu, z \rangle > 0\} \quad \text{and} \quad \Phi^- := \{\nu \in \Phi : \langle \nu, z \rangle < 0\}$$

where $z \in C_\Sigma$ is an arbitrary vector in the chamber.

Further background on reflection groups and proofs of statements made here can be found in, for example, Kane (2001); Humphreys (1990). Specific examples will be introduced in due course.

1.2. *Brownian motion in* C_Σ. Suppose that we have chosen a root system Φ, fixed a set of fundamental roots Σ and hence determined a fundamental chamber C_Σ with which to work. We are now able to consider Brownian motion in this domain. Let $(B_t, t \geq 0)$ be a standard Brownian motion in V and denote the law of B started at $x \in V$ by $\mathbb{P}_x$. Recall that the transition density of Brownian motion in V is given by

$$p_t(x,y) = \frac{1}{Z_t} \exp\left(- \frac{\langle y-x\,,\, y-x\rangle}{2t}\right) \qquad x, y \in V \tag{1.3}$$

where Z_t is a normalisation constant. Note that p_t is W-invariant, that is, $p_t(wx, wy) = p_t(x,y)$ for all $w \in W$.

Let $T_{C_\Sigma} := \inf\{t \geq 0 : B_t \notin C_\Sigma\}$ be the first exit time of B from C_Σ. and write $\hat{p}_t(x,y)$ for the transition density of Brownian motion in C_Σ, killed on reaching the boundary $\partial C_\Sigma = \overline{C}_\Sigma \setminus C_\Sigma$. Recall that $\hat{p}_t$ is the heat kernel on C_Σ with Dirichlet conditions on the boundary. A generalisation of the classical reflection principle states that (see, for example, Biane (1992); Gessel and Zeilberger (1992); Grabiner (1999); Karlin and McGregor (1959))

$$\hat{p}_t(x,y) = \sum_{w \in W(\Phi)} \varepsilon(w)\, p_t(x, wy), \tag{1.4}$$

for $x, y \in \overline{C}_\Sigma$, where

$$\varepsilon(w) = \det(w) = \begin{cases} 1 & \text{if } w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k} \text{ and k is even.} \\ -1 & \text{if } w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k} \text{ and k is odd.} \end{cases}$$

We include a proof here for completeness which is valid for all root systems. Fix a Borel subset A of $\overline{C}_\Sigma$. Since

$$\int_A \hat{p}_t(x,y)\, \mathrm{d}y = \mathbb{P}_x(B_t \in A; T_{C_\Sigma} > t),$$

it suffices to show that

$$\mathbb{P}_x(B_t \in A; T_{C_\Sigma} > t) = \sum_{w \in W(\Phi)} \varepsilon(w) \mathbb{P}_x(B_t \in wA)$$

or, equivalently,

$$\sum_{w \in W(\Phi)} \varepsilon(w) \mathbb{P}_x(B_t \in wA; T_{C_\Sigma} \le t) = 0.$$

For $\alpha \in \Sigma$, set $T_\alpha = \inf\{t \ge 0 : \langle B_t, \alpha \rangle = 0\}$. Then $T_{C_\Sigma} = \min_{\alpha \in \Sigma} T_\alpha$ and

$$\begin{aligned} &\sum_{w \in W(\Phi)} \varepsilon(w) \mathbb{P}_x(B_t \in wA; T_{C_\Sigma} \le t) \\ &\quad = \sum_{\alpha \in \Sigma} \sum_{w \in W(\Phi)} \varepsilon(w) \mathbb{P}_x(B_t \in wA; T_{C_\Sigma} = T_\alpha \le t). \end{aligned}$$

It therefore suffices to show that, for each $\alpha \in \Sigma$,

$$\sum_{w \in W(\Phi)} \varepsilon(w) \mathbb{P}_x(B_t \in wA; T_{C_\Sigma} = T_\alpha \le t) = 0.$$

Fix α, and define $\hat{B}_u = B_u 1_{u \le T_\alpha} + s_\alpha B_u 1_{u > T_\alpha}$. By the strong Markov property and W-invariance of p_t, $\hat{B}$ has the same law as B. It follows that

$$\begin{aligned} \mathbb{P}_x(B_t \in wA; T_{C_\Sigma} = T_\alpha \le t) &= \mathbb{P}_x(\hat{B}_t \in s_\alpha wA; T_{C_\Sigma} = T_\alpha \le t) \\ &= \mathbb{P}_x(B_t \in s_\alpha wA; T_{C_\Sigma} = T_\alpha \le t), \end{aligned}$$

and so

$$\begin{aligned} &\sum_{w \in W(\Phi)} \varepsilon(w) \mathbb{P}_x(B_t \in wA; T_{C_\Sigma} = T_\alpha \le t) \\ &\quad = \sum_{w \in W(\Phi)} \varepsilon(w) \mathbb{P}_x(B_t \in s_\alpha wA; T_{C_\Sigma} = T_\alpha \le t) \\ &\quad = -\sum_{w \in W(\Phi)} \varepsilon(s_\alpha w) \mathbb{P}_x(B_t \in s_\alpha wA; T_{C_\Sigma} = T_\alpha \le t) \\ &\quad = -\sum_{w \in W(\Phi)} \varepsilon(w) \mathbb{P}_x(B_t \in wA; T_{C_\Sigma} = T_\alpha \le t), \end{aligned}$$

as required.

Brownian motion killed at the boundary of the chamber can be conditioned to remain in C_Σ by use of a Doob h-transform. An introductory account of h-transforms is given in Rogers and Williams (2000). Broadly speaking, a h-transform is a tool that may be used to condition a process to converge to a specific point. In the case we consider here, the theory is applied to ensure that the space-time Brownian motion (B_t, t) exits the domain C_Σ at $t = \infty$ i.e. it remains alive. The appropriate function for this transformation (see Biane (1994)) is the unique, up to a scaling factor, positive $\hat{p}$-harmonic function h on $\overline{C}_\Sigma$ which (crucially) vanishes on the boundary of the chamber. This function is given by

$$h(x) := \prod_{\alpha \in \Phi^+} \langle \alpha, x \rangle \qquad x \in V \tag{1.5}$$

This product is sometimes called the alternating polynomial or discriminant associated with Φ (see Dunkl and Xu (2001); Humphreys (1990)).

Now a Brownian motion started at $x \in C_\Sigma$, conditioned to stay in C_Σ, is, by definition, the corresponding h-process with law denoted $\mathbb{Q}_x$ and transition density given by

$$q_t(x,y) := \frac{h(y)}{h(x)} \hat{p}_t(x,y) \qquad x \in C_\Sigma \tag{1.6}$$

We can interpret $\frac{h(y)}{h(x)}$ as $\frac{\mathbb{P}_y[T_{C_\Sigma}=\infty]}{\mathbb{P}_x[T_{C_\Sigma}=\infty]}$ which aligns (1.6) with what we would intuitively expect. In fact it has been shown, at varying levels of generality (refer to e.g. Grabiner (1999); König and O'Connell (2001); Doumerc and O'Connell (2005)) that $\mathbb{P}_x\left[T_{C_\Sigma} > t\right] \sim h(x) \frac{c}{t^{\frac{m}{2}}}$ as $t \to \infty$ where c is a constant and $m = |\Phi^+|$.

Note that the above density (1.6) is defined for starting points actually in the chamber not in its closure. Thus in order to consider Brownian motion conditioned to remain in C_Σ but started from a point on the boundary we would need to take an appropriate limit, which is possible for all points on the boundary ∂C_Σ by continuity.

It can be shown directly or by use of the operateur carré du champ (see later) that the generator of the transformed process is

$$\mathscr{L}^h = \frac{1}{2}\Delta + \langle \nabla \log h, \nabla \rangle \tag{1.7}$$

This operator also arises naturally (see, for example, Gallardo and Yor (2005)) in the context of $W(\Phi)$-radial Dunkl processes.

1.3. *Examples.* We present three examples of root systems whose fundamental chambers have natural interpretations in terms of the coordinates of n-dimensional Brownian motion. Here we set $V = \mathbb{R}^n$ and let $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ denote the standard orthonormal basis of $\mathbb{R}^n$. It is easily checked that the given values satisfy the necessary requirements.

Example 1: "Type A"

We have

$$\begin{aligned} \Phi &= \{\epsilon_i - \epsilon_j \,|\, i \neq j\} \\ \Phi^+ &= \{\epsilon_i - \epsilon_j \,|\, i < j\} \\ \Sigma &= \{\epsilon_i - \epsilon_{i+1} \,|\, i < n-1\} \end{aligned}$$

The effect of $s_{\epsilon_i-\epsilon_j}$ on $x \in \mathbb{R}^n$ is to interchange the i'th and j'th coordinates. Since the symmetric group S_n, which acts on $x \in \mathbb{R}^n$ by permuting coordinates, is generated by transpositions, we can write

$$W(\Phi) = S_n$$

The corresponding fundamental chamber is

$$C_\Sigma = \{x \in \mathbb{R}^n : x_1 > x_2 > \cdots > x_n\}$$

Therefore, an n-dimensional Brownian motion conditioned to stay in C_Σ is conditioned for its coordinates, the n one dimensional Brownian motions, to never coincide. In other words, this provides a description of a system of n one-dimensional Brownian particles conditioned to never collide. The alternating polynomial is

$$h(x) = \prod_{1 \leq i < j \leq n} x_i - x_j$$

which is otherwise known as Vandermonde's determinant.

¿From (1.7) we see that an n-dimensional Brownian motion conditioned to stay in the fundamental chamber corresponding to the type A root system, gives rise to the diffusion

$$\mathrm{d}\lambda_t^{(i)} = \mathrm{d}\hat{\beta}_t^{(i)} + \sum_{j \neq i} \frac{1}{\lambda_t^{(i)} - \lambda_t^{(j)}}\,\mathrm{d}t \qquad i = 1, \ldots, n. \tag{1.8}$$

where $(\hat{\beta}_t, t \geq 0)$ is a standard Brownian motion in $\mathbb{R}^n$.

Recall that a possible construction of an $n \times n$ GUE matrix M is to let

$$\begin{aligned} m_{jj} &= \eta_{jj} \quad j = 1, \ldots, n \\ m_{jk} &= \eta_{jk}^{(r)} + i\eta_{jk}^{(i)} \quad 1 \leq j < k \\ m_{jk} &= \overline{m}_{kj} \quad k < j \leq n \quad \text{with } \overline{a+ib} = a - ib \end{aligned}$$

where

$$\eta_{jj} \sim \text{Normal}(0,1) \quad \text{and} \quad \eta_{jk}^{(r)}, \eta_{jk}^{(i)} \sim \text{Normal}(0, 1/2)$$

are independent random variables. The process version is obtained by replacing the normal random variables with suitably scaled Brownian motions.

In Dyson (1962) Dyson observes that the SDE (1.8), with initial condition $\lambda_0 = (0, \ldots, 0)$, is satisfied by the eigenvalues of the process version of GUE (see also Mehta (2004)). For this reason the above diffusion is often referred to as Dyson's Brownian motion. We have (see König and O'Connell (2001))

$$\lim_{x \to 0\,:\, x \in C_\Sigma} q_t(x, y) = \frac{1}{\prod_{j=0}^{n-1} t^j j!} h(y)^2 p_t(0, y) \tag{1.9}$$

which for $t = 1$ corresponds to the joint density of the eigenvalues of an $n \times n$ GUE matrix.

Example 2: "Type B"

In this case

$$\begin{aligned} \Phi &= \{\epsilon_i - \epsilon_j \,|\, i \neq j\} \cup \{v_{i,j} := \text{sign}(j-i)(\epsilon_i + \epsilon_j) \,|\, i \neq j\} \cup \{\pm\epsilon_i\} \\ \Phi^+ &= \{\epsilon_i - \epsilon_j\,,\, \epsilon_i + \epsilon_j \,|\, i < j\} \cup \{\epsilon_i \,:\, 1 \leq i \leq n\} \\ \Sigma &= \{\epsilon_i - \epsilon_{i+1} \,|\, i < n-1\} \cup \{\epsilon_n\} \end{aligned}$$

Now $s_{\epsilon_i-\epsilon_j}$ (as before) switches the i'th and j'th coordinates, $s_{v_{i,j}}$ switches the signs of the i'th and j'th coordinates and s_{ϵ_i} changes the sign of the i'th coordinate. $W(\Phi)$ is the symmetry group of the hyperoctahedron. The corresponding fundamental chamber is

$$C_\Sigma = \{x \in \mathbb{R}^n \; : \; x_1 > x_2 > \cdots > x_n > 0\}$$

An n-dimensional Brownian motion in this chamber is interpreted as a system of n one-dimensional Brownian motions conditioned to never collide and to remain positive. The alternating polynomial is

$$h(x) = \prod_{i=1}^{n} x_i \prod_{1 \le i < j \le n} x_i^2 - x_j^2$$

Example 3: "Type D"

This root system is a subset of the "type B" root system. In this case

$$\begin{aligned} \Phi &= \{\epsilon_i - \epsilon_j \,|\, i \neq j\} \cup \{v_{i,j} := \operatorname{sign}(j-i)(\epsilon_i + \epsilon_j) \,|\, i \neq j\} \\ \Phi^+ &= \{\epsilon_i - \epsilon_j \,,\, \epsilon_i + \epsilon_j \,|\, i < j\} \\ \Sigma &= \{\epsilon_i - \epsilon_{i+1} \,|\, i < n-1\} \cup \{\epsilon_{n-1} + \epsilon_n\} \end{aligned}$$

This gives the fundamental chamber

$$C_\Sigma = \{x \in \mathbb{R}^n \; : \; x_1 > x_2 > \cdots > x_{n-1} > |x_n|\}$$

Brownian motion conditioned to stay in this chamber can be thought of, if we ignore the sign of the last coordinate, as a system of n one-dimensional reflected Brownian motions conditioned never to collide. The alternating polynomial is given by

$$h(x) = \prod_{1 \le i < j \le n} x_i^2 - x_j^2$$

These three root system examples are "classical" in the sense that they are associated with classical semisimple Lie algebras. A root system Φ qualifies as a root system in the Lie theory sense if, in addition to the axioms R1 and R2 stated at the beginning of this section, Φ spans V (said to be essential) and we have $\frac{2\langle \nu,\alpha \rangle}{\langle \alpha,\alpha \rangle} \in \mathbb{Z} \; \forall \; \alpha, \nu \in \Phi$ (it is crystallographic). $W(\Phi)$ is then a Weyl group and its fundamental chambers are Weyl chambers. It transpires that such root systems are enough to completely determine their associated Lie algebras. This will be discussed further in the third section. Additional examples and explanation may be found in, for example, Kane (2001); Humphreys (1990, 1978).

2. Adding a drift

2.1. *Other processes in* C_Σ. The presentation has so far been concentrated on Brownian motion but these constructions are valid for a slightly more general class of processes. The argument justifying the form of the transition probability (1.4) is applicable to other Markov processes that are "reflectable" in the sense that their state spaces and their laws are invariant under the action of the reflection group. In these cases the transition density would easily be adapted by substituting in the relevant density for p_t. There is a fuller discussion of this in Grabiner (1999).

Clearly the h-transform (1.5) will not have the same effect for all processes. The function used for the h-transformation needs to be harmonic with respect to the process being conditioned. If the original process has generator $\mathscr{L}$ we require $\mathscr{L}h = 0$ on C_Σ. Examples of processes for which the alternating polynomial is still applicable in this context are provided in König and O'Connell (2001) for the "type A" root system.

2.2. *Brownian motion with drift in* C_Σ. We now turn our attention to Brownian motion with drift. Recall that the transition density of a Brownian motion in V with drift μ is given by

$$p_t^\mu(x,y) := \frac{1}{Z_t} \exp\Big[-\frac{\langle y-x-\mu t, y-x-\mu t\rangle}{2t}\Big] \qquad x,y,\mu \in V \tag{2.1}$$

We will be particularly interested in cases for which the drift coordinates are not equal but remain the same over time. In general, Brownian motion with drift is "reflectable" only when the coordinates of μ are equal. However, as is well known, the law $\mathbb{P}_x^\mu$ (on the canonical space) of Brownian motion with drift μ, started at x and the law $\mathbb{P}_x$ of standard Brownian motion started at x are equivalent on the natural filtration $\{\mathscr{F}_t^0\}$ with respect to the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{P}_x^\mu}{\mathrm{d}\mathbb{P}_x}\bigg|_{\mathscr{F}_t^0} = \exp\Big[\langle y-x,\mu\rangle - \frac{\langle \mu,\mu t\rangle}{2}\Big]$$

We can therefore deduce that the transition probability of a Brownian motion with drift $\mu \in V$, started at $x \in C_\Sigma$, killed at the boundary of C_Σ is

$$\begin{aligned}
\hat{p}_t^\mu(x,y)\mathrm{d}y &:= \mathbb{P}_x^\mu[B_t \in \mathrm{d}y\,;\, T_{C_\Sigma} > t] \\
&= \mathbb{P}_x[B_t \in \mathrm{d}y\,;\, T_{C_\Sigma} > t] \exp\Big[\langle y-x,\mu\rangle - \frac{\langle \mu,\mu t\rangle}{2}\Big] \\
&= \sum_{w\in W(\Phi)} \varepsilon(w)\, p_t(x,wy) \exp\Big[\langle y-x,\mu\rangle - \frac{\langle \mu,\mu t\rangle}{2}\Big]\,\mathrm{d}y
\end{aligned} \tag{2.2}$$

We will now focus on the cases with drift vector $\mu \in C_\Sigma$. We present a slightly modified approach to that discussed in the last section, which allows us to condition a Brownian motion with drift $\mu \in C_\Sigma$ to remain in a fundamental chamber. Let $\mathbb{Q}_x^\mu$ be the law under which a Brownian motion with drift $\mu \in C_\Sigma$ started at $x \in C_\Sigma$ stays in C_Σ. Analogously to the non-drifting case, we would expect

$$\mathbb{Q}_x^\mu[B_t \in dy] = \mathbb{P}_x^\mu[B_t \in dy\,;\, T_{C_\Sigma} > t]\, \frac{\mathbb{P}_y^\mu[T_{C_\Sigma} = \infty]}{\mathbb{P}_x^\mu[T_{C_\Sigma} = \infty]} \tag{2.3}$$

We need to ensure that this makes sense.

Lemma 2.1. *The probability that a Brownian motion with drift $\mu \in C_\Sigma$ started at $x \in C_\Sigma$ never exits the chamber is given by*

$$\begin{aligned}
\mathbb{P}_x^\mu[T_{C_\Sigma} = \infty] &:= \lim_{t\to\infty} \mathbb{P}_x^\mu[T_{C_\Sigma} > t] \\
&= \sum_{w\in W(\Phi)} \varepsilon(w) \exp[\langle \mu, wx - x\rangle]
\end{aligned}$$

Henceforth, we shall denote this function by

$$h_\mu(x) := \sum_{w\in W(\Phi)} \varepsilon(w)\exp[\langle \mu, wx - x\rangle] \tag{2.4}$$

Proof. As laid out in Biane et al. (2005), the exit time distribution is given by

$$\begin{aligned}\mathbb{P}^\mu_x[T_{C_\Sigma} > t] &= \int_{C_\Sigma} \mathbb{P}^\mu_x[B_t \in \mathrm{d}y; T_{C_\Sigma} > t]\,\mathrm{d}y \\ &= \int_{C_\Sigma} \sum_{w\in W(\Phi)} \varepsilon(w) p_t(x, wy)\exp\left[\langle \mu, y - x\rangle - \frac{\langle \mu, \mu t\rangle}{2}\right]\mathrm{d}y\end{aligned}$$

Since p_t is invariant under the action of $w \in W(\Phi)$ we have

$$p_t(x, wy) = p_t(0, y - wx)$$

so

$$\mathbb{P}^\mu_x[T_{C_\Sigma} > t] = \sum_{w\in W(\Phi)} \varepsilon(w)\int_{C_\Sigma} p_t(0, y - wx)\exp\left[\langle \mu, y - x\rangle - \frac{\langle \mu, \mu t\rangle}{2}\right]\mathrm{d}y$$

Now observe that

$$\int_V p_t(0, y - x)\exp\left[\langle \mu, y - x\rangle - \frac{\langle \mu, \mu t\rangle}{2}\right]\mathrm{d}y = 1$$

However, since $\mu \in C_\Sigma$ we have that as $t \to \infty$

$$\int_{V\setminus C_\Sigma} \exp\left[\langle \mu, y - x\rangle - \frac{\langle \mu, \mu t\rangle}{2}\right]\mathrm{d}y \to 0$$

Hence

$$\begin{aligned}&\int_{C_\Sigma} p_t(0, y - wx)\exp\left[\langle \mu, y - x + wx - wx\rangle - \frac{\langle \mu, \mu t\rangle}{2}\right]\mathrm{d}y \\ =\ & \exp[\langle \mu, wx - x\rangle]\int_{C_\Sigma} p_t(0, y - wx)\exp\left[\langle \mu, y - wx\rangle - \frac{\langle \mu, \mu t\rangle}{2}\right]\mathrm{d}y \\ \to\ & \exp[\langle \mu, wx - x\rangle] \qquad \text{as } t \to \infty.\end{aligned}$$

Substituting this limit into the above expression for the exit time distribution gives (2.4). □

Note that

$$\mathbb{E}_{\mathbb{P}^\mu_x}[h_\mu(B_t)] = \sum_{w\in W(\Phi)} \varepsilon(w)\,\mathbb{E}_{\mathbb{P}_x}\left[\exp\left[\langle \mu, wB_t - x\rangle - \frac{\langle \mu, \mu t\rangle}{2}\right]\right]$$

Since standard Brownian motion is invariant under orthogonal transformations (i.e. wB_t is still a Brownian motion under $\mathbb{P}_x$) it follows that $h_\mu(B_t)$ is an $(\mathscr{F}^0_t, \mathbb{P}^\mu_x)$ - continuous martingale for each x.

Now Girsanov's theorem and the fact that T_{C_Σ} is an $\{\mathscr{F}^0_t\}$ - stopping time gives us the result that for every x and $t > 0$ we can define a measure $\mathbb{Q}^\mu_x$ equivalent to $\mathbb{P}^\mu_x$ on $\mathscr{F}^0_t$ with respect to the Radon-Nikodym derivative

$$\left.\frac{\mathrm{d}\mathbb{Q}^\mu_x}{\mathrm{d}\mathbb{P}^\mu_x}\right|_{\mathscr{F}^0_t} = \frac{h_\mu(B_{t\wedge T_{C_\Sigma}})}{h_\mu(x)} \tag{2.5}$$

Consequently, we can write

$$\mathbb{Q}_x^\mu[B_t \in \mathrm{d}y] = \mathbb{P}_x^\mu[B_t \in \mathrm{d}y\,;\, T_{C_\Sigma} > t]\,\frac{h_\mu(y)}{h_\mu(x)} \tag{2.6}$$

which agrees with our "heuristic" (2.3). The corresponding transition density, the transition density for a Brownian motion with drift $\mu \in C_\Sigma$ started at $x \in C_\Sigma$, killed at the boundary of the chamber, conditioned on $\{T_{C_\Sigma} = \infty\}$, is given by

$$q_t^\mu(x,y) := \hat{p}_t^\mu(x,y)\,\frac{h_\mu(y)}{h_\mu(x)} \tag{2.7}$$

Lemma 2.2. *The conditioned process with transition density $q_t^\mu(x,y)$ is a diffusion in C_Σ with (extended) infinitesimal generator equal to*

$$\mathscr{L}^{h_\mu} = \frac{1}{2}\Delta + \big\langle \mu, \nabla \big\rangle + \big\langle \nabla \log h_\mu, \nabla \big\rangle \tag{2.8}$$

on C^2.

Proof. Our construction fits the framework (as described in Revuz and Yor (1999)) of the operateur carré du champ method of obtaining the generator of a transformed process from that of the original. The operateur carré du champ is given by

$$\Theta(f,g) := \mathscr{L}(fg) - f\mathscr{L}g - g\mathscr{L}f \tag{2.9}$$

for $f, g \in C^2$. Let $\mathscr{L}$ be the (extended) generator of a diffusion process X with law $\widetilde{\mathbb{P}}_x$ and define

$$D_t := \exp\big[f(X_t) - f(X_0) - \int_0^t F(X_s)\,\mathrm{d}s\big]$$

If D_t is a $\widetilde{\mathbb{P}}_x$-continuous martingale then the extended generator of the process corresponding to X under the law $\mathbb{Q}_{x,f} := D_t \cdot \widetilde{\mathbb{P}}_x$ is equal to

$$\mathscr{L} + \Theta(f, \cdot)$$

on C^2. In this particular case we have

$$\begin{aligned} \mathscr{L} &= \frac{1}{2}\Delta + \big\langle \mu, \nabla \big\rangle \\ f &= \log h_\mu \\ F &= 0 \end{aligned} \tag{2.10}$$

The result now follows from a straightforward computation. □

Note that alternatively, we can write the generator (2.8) as

$$\mathscr{L}^{h_\mu} = \frac{1}{2}\Delta + \big\langle \nabla \log \varphi_\mu, \nabla \big\rangle \tag{2.11}$$

where

$$\varphi_\mu(x) := \sum_{w \in W(\Phi)} \varepsilon(w) \exp\Big[\big\langle \mu, wx \big\rangle\Big]$$

Proposition 2.3. *We claim that*

a) *The transition densities extend by continuity to the boundary of C_Σ. For the process started at the origin we specifically have*

$$\begin{aligned} q_t^\mu(0,y) &:= \lim_{\substack{x\to 0\\ x\in C_\Sigma}} q_t^\mu(x,y) \\ &= h_\mu(y)\,\frac{h(y/t)}{h(\mu)}\,p_t^\mu(0,y) \end{aligned} \tag{2.12}$$

b) *Further, for $t=1$, the transition densities of*
   i) *a Brownian motion with drift $\mu \in C_\Sigma$, started at $x \in C_\Sigma$, conditioned to remain in the chamber*

   and

   ii) *a Brownian motion with drift $x \in C_\Sigma$, started at $\mu \in C_\Sigma$, conditioned to remain in the chamber*

are the same.

That is

$$q_1^\mu(x,y) = q_1^x(\mu,y) \tag{2.13}$$

On combining statements a) and b) we get

$$q_1(\mu,y) = q_1^\mu(0,y) = h_\mu(y)\,\frac{h(y)}{h(\mu)}\,p_1(\mu,y) \tag{2.14}$$

Proof. ¿From (2.7) we know

$$q_t^\mu(x,y) := \frac{h_\mu(y)}{h_\mu(x)}\,\hat{p}_t(x,y)\,\exp\Big[\langle y-x,\mu\rangle - \frac{\langle\mu,\mu\rangle t}{2}\Big] \tag{2.15}$$

Now

$$\begin{aligned} &\hat{p}_t(x,y) \\ = {}& \frac{1}{Z_t}\sum_{w\in W(\Phi)} \varepsilon(w)\,\exp\Big[-\frac{\langle wy-x, wy-x\rangle}{2t}\Big] \\ = {}& \frac{1}{Z_t} e^{-<y-x,y-x>/2t} \sum_{w\in W(\Phi)} \varepsilon(w) e^{<w(y/t)-(y/t)\,,x>}\exp\Big[-\frac{<wy-y, wy+y>}{2t}\Big] \\ = {}& p_t(x,y)h_x(y/t) \end{aligned}$$

where for the last step we have used the fact that w preserves the inner product. By putting this expression for $\hat{p}_t$ back into (2.15) we get

$$q_t^\mu(x,y) := \frac{h_\mu(y)}{h_\mu(x)}\,h_x(y/t)p_t^\mu(x,y) \tag{2.16}$$

Since h_μ involves summation over the whole of $W(\Phi)$ and the inner product is symmetric we have $h_\mu(x) = h_x(\mu)$. Finally we note that

$$\frac{h_x(y/t)}{h_x(\mu)} \longrightarrow \frac{h(y/t)}{h(\mu)} \qquad \text{as (the drift) } x\to 0$$

Thus

$$\begin{aligned} q_t^\mu(0,y) &:= \lim_{x\to 0} q_t^\mu(x,y) \\ &= \lim_{x\to 0} h_\mu(y)\,\frac{h_x(y/t)}{h_x(\mu)}\,p_t^\mu(x,y) \\ &= h_\mu(y)\frac{h(y/t)}{h(x)}p_t^\mu(0,y) \end{aligned}$$

as required.

If we now write (2.16) with $t=1$ and again use the interchangeability of $h_\mu(x)$ and $h_x(\mu)$ we find

$$\begin{aligned} q_1^\mu(x,y) &:= \frac{h_\mu(y)}{h_\mu(x)}\,h_x(y)p_1^\mu(x,y) \\ &= \frac{h_\mu(y)}{h_x(\mu)}\,h_x(y)p_1^x(\mu,y) \\ &= q_1^x(\mu,y) \end{aligned}$$

□

3. A Special Drift Vector

We will now consider the special case of when the drift vector of our Brownian motion is given by

$$\mu = \rho := \sum_{\alpha\in\Phi^+} \alpha \tag{3.1}$$

where Φ^+ denotes the set of positive roots determined by our chosen set of fundamental roots Σ. We have $\rho \in C_\Sigma$ since $\langle \rho, \alpha_i\rangle > 0 \ \forall\, \alpha_i \in \Sigma$. This can be seen by noting that the effect of s_{α_i}, $\alpha_i \in \Sigma$ is to permute the elements of $\Phi^+ \setminus \{\alpha_i\}$ and to send α_i to its negative. Thus $s_{\alpha_i}\rho = \rho - 2\alpha_i$ which implies

$$2\frac{\langle \alpha_i, \rho\rangle}{\langle \alpha_i, \alpha_i\rangle} = 2$$

In the language of representation theory this makes ρ a dominant weight. Refer to Humphreys (1978) for further details.

¿From Weyl's denominator identity (see e.g. Kane (2001); Hall (2003)) it follows that

$$\begin{aligned} \varphi_\rho(x) &:= \sum_{w\in W(\Phi)} \varepsilon(w)e^{<\rho,wx>} \\ &= \prod_{\alpha\in\Phi^+} e^{<\alpha,x>} - e^{-<\alpha,x>} \\ &= \prod_{\alpha\in\Phi^+} 2\,\sinh\langle \alpha, x\rangle \end{aligned} \tag{3.2}$$

¿From (2.11) we have that the generator of a Brownian motion with drift ρ conditioned to remain in C_Σ is

$$\mathscr{L}^{h_\rho} = \frac{1}{2}\Delta + \langle \nabla \log\varphi_\rho, \nabla\rangle \tag{3.3}$$

We remark that if $W(\Phi)$ is a Weyl group, this operator, multiplied by two (as is conventional in differential geometry) and written as

$$\Delta + 2\langle \nabla \log \delta^{\frac{1}{2}}, \nabla \rangle \tag{3.4}$$

where

$$\delta(x) = \prod_{\alpha \in \Phi+} \sinh^2 \langle \alpha, x \rangle \tag{3.5}$$

arises in the context of symmetric spaces associated with complex semisimple Lie groups (Helgason (1984) II.3). In order to clarify what we mean by this statement we will need to refer to the following framework. Further details and definitions of the terminology used but not defined here, can be found in e.g Hall (2003); Helgason (1978) .

3.1. *The symmetric space setting.* We will consider a non-compact symmetric space P of the form G/K. G will be a complex semisimple non-compact connected Lie group with finite centre and Lie algebra $\mathfrak{g}$ and K a maximal compact subgroup of G with Lie algebra $\mathfrak{l}$. We let

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$$

denote the Cartan decomposition of $\mathfrak{g}$ and let $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p}$. We use the Killing form on $\mathfrak{g}$ restricted to $\mathfrak{a}$ to identify elements of $\mathfrak{a}$ with linear functionals $\alpha : \mathfrak{a} \to \mathbb{C}$ in its dual space $\mathfrak{a}^*$. We then let

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : [x, H] = \alpha(H)x \quad H \in \mathfrak{a}\}$$

where [,] denotes the Lie bracket. This enables us to decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{a}^*} \mathfrak{g}_\alpha \tag{3.6}$$

This decomposition can be refined to give the root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha = \mathfrak{l} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Phi+} \mathfrak{g}_\alpha$$

This $\Phi \subset \mathfrak{a}^*$ is called the root system of $(\mathfrak{g}, \mathfrak{a})$. We can find an Euclidean space V of the same dimension as $\mathfrak{a}^*$ such that $\Phi \subset V \subset \mathfrak{a}^*$ and then Φ is an essential, crystallographic root system in the same sense as in previous sections. Now we have similar notions of positive roots Φ^+ and fundamental roots Σ as before. We define the Weyl chamber

$$\mathfrak{a}^+ := \{H \in \mathfrak{a} \, : \, \alpha_i(H) > 0 \; \forall \, \alpha_i \in \Sigma\} \tag{3.7}$$

and note that though $\mathfrak{a}^+$ is not the same as C_Σ there is a natural correspondence between them. Since there is a one-to-one relation between essential, crystalographic root systems and complex semisimple Lie algebras we can actually start with a given root system or Weyl chamber and work backwards to identify the associated symmetric space.

We will be paying close attention to the polar decomposition on P as defined in Helgason (1978) Ch IX Corollary 1.2. For any $x \in P$ we have

$$x = K(x) \exp \Gamma(x) \cdot o \tag{3.8}$$

where $K(x) \in K$ and o is the identity coset or the origin in P. $\Gamma(x)$, referred to as the generalised radial component, is an element of $\bar{\mathfrak{a}}^+$ and is uniquely determined.

Brownian motion $\widetilde{B}$ on the symmetric space P is defined as the Markov process with generator equal to $\frac{\Delta_P}{2}$ where Δ_P is the Laplace-Beltrami operator on P. An introduction to this topic may be found in Gangolli (1965), Applebaum (2000) and Taylor (1988). The Laplace-Beltrami operator on P can be expressed as a sum of two elliptic operators corresponding to the so-called radial and angular parts associated with the polar decomposition on P. Refer to Helgason (1984) Ch II Theorem 5.24 and Theorem 3.7 for the relevant general statements and Taylor (1991) for further details in this specific context. Incidentally, this is a specific example of a skew-product decomposition as discussed in Pauwels and Rogers (1988).

Here, we are interested in the radial part of Δ_P viewed as an operator on $f \in C^\infty(\mathfrak{a}^+)$ and given by

$$\mathrm{Rad}(\Delta_P) = \Delta_{\mathfrak{a}} + 2\left\langle \nabla_{\mathfrak{a}} \log \delta^{\frac{1}{2}}, \nabla_{\mathfrak{a}} \right\rangle \tag{3.9}$$

Note that this is equal to the operator (3.4) when Δ and ∇ are regarded as the Laplacian and gradient operators on $\mathfrak{a}$.

As explained in the appendix of Taylor (1991), Brownian motion will almost surely only visit the regular points of P, those points that are decomposed in terms of $\mathfrak{a}^+$ (rather than $\bar{\mathfrak{a}}^+ \setminus \mathfrak{a}^+$), since the non-regular points have capacity zero. Consequently, the radial component $\Gamma(\widetilde{B})$ of $\widetilde{B}$ on P started at $o = \{K\}$ remains in $\mathfrak{a}^+$ for all time and has generator $\frac{1}{2}\mathrm{Rad}(\Delta_P)$ (see Orihara (1970), Taylor (1991)).

We suppose that we give $\mathfrak{a}$ an Euclidean structure using, for example, the Killing form as an inner product. The following observation now follows from the above discussion.

Proposition 3.1. *Euclidean Brownian motion on* $\mathfrak{a}$ *with drift* $\rho := \sum_{\alpha\in\Phi^+} \alpha$, *started at the origin and conditioned to remain in the Weyl chamber* $\mathfrak{a}^+$, *has the same distribution as* $\Gamma(\widetilde{B})$ *the radial part of the Brownian motion* $\widetilde{B}$ *on the corresponding (as formulated above) symmetric space* G/K.

Discussion of other radial parts associated with standard driftless Brownian motion in a Weyl chamber can be found in Anker et al. (2002) and Bougerol and Jeulin (2002). The second of these papers utilises the above framework to provide generalisations of and geometric interpretations for certain path transformations associated with random matrix theory. Specifically, they show that if X is an Euclidean Brownian motion on $\mathfrak{p}$ and $\hat{B}$ is a standard Euclidean Brownian motion on $\mathfrak{a}$ then the radial component $\mathrm{rad}(X)$ (the counterpart of Γ on the algebra level), has the same distribution as $\hat{B}$ conditioned to remain in the Weyl chamber $\mathfrak{a}^+$. For certain cases they prove that $\mathrm{rad}(X)$ can be realised as a continuous path transformation of $\hat{B}$ and conjecture that such a result holds more generally. This was recently confirmed in Biane et al. (2005). Since the radial part of standard Brownian motion is well known to be the Bessel process, they refer to the radial process $\mathrm{rad}(X)$ as a generalised Bessel process. This is also discussed in Grabiner (1999). The observation of Proposition 3.1 can be viewed as an analogous relationship between the Brownian motion on the non-compact symmetric space and the Euclidean Brownian motion on $\mathfrak{a}$ with this special drift.

3.2. *The interpretation for the type A case.* In order to illustrate what the above technical description means in practice, we consider a specific example. An elementary account of Brownian motion on the symmetric space $SL(2;\mathbb{R})/SO(2)$, explained with minimal reference to differential geometry, can be found in Rogers and Williams (1987) V.36. This is a simplification of an account of the related topic of Brownian motions of ellipsoids, i.e. Brownian motion on the space of positive definite symmetric matrices, given in Norris et al. (1986). Of course, as we have specified we are concerned with the non-compact symmetric spaces associated with the complex rather than real semisimple Lie groups. Nevertheless, as we will see, the arguments and outcomes presented in these references are very similar to the complex analogues that we consider here and consequently our exposition, will, in parts, have much in common with theirs.

We will look in detail at the "type A" case. This will be of further interest in the final part of this work. With notation as above, the complex semisimple Lie algebra associated with the "type A" root system (see Helgason (1978) Ch III) is

$$\mathfrak{g} = \mathfrak{sl}(n;\mathbb{C}) := \{n \times n \,\text{complex matrices with zero trace}\}$$

This is the Lie algebra of the complex, semisimple, non-compact, connected Lie group

$$G = SL(n;\mathbb{C}) := \{n \times n \,\text{complex invertible matrices with determinant one}\}$$

The maximal compact subgroup of $G = SL(n;\mathbb{C})$ is

$$K = SU(n) := \{n \times n \,\text{unitary matrices with determinant one}\}$$

Therefore, the symmetric space associated with the "type A" root system is

$$P := G/K = SL(n;\mathbb{C})/SU(n)$$

It can be checked by considering the cosets of $SU(n)$ in $SL(n;\mathbb{C})$ that P is identified with the space of $n \times n$ positive definite Hermitian matrices of determinant one.

Now $K = SU(n)$ has Lie algebra

$$\mathfrak{k} = \mathfrak{su}(n) := \{n \times n \,\text{skew Hermitian matrices}\}$$

and the corresponding Cartan decomposition is given by

$$\mathfrak{sl}(n;\mathbb{C}) = \mathfrak{su}(n) \oplus \mathfrak{p}$$

where

$$\mathfrak{p} := \{n \times n \text{ Hermitian matrices with zero trace }\}$$

We have

$$\mathfrak{a} := \{\mathrm{diag}(\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(n)}) \,:\, \gamma^{(1)} + \cdots + \gamma^{(n)} = 0\}$$

as a maximal Abelian subspace of $\mathfrak{p}$. The linear functionals on $H \in \mathfrak{a}$ given by

$$\begin{aligned} \alpha(H) &= \gamma^{(i)} - \gamma^{(j)} \qquad && i \neq j \quad i,j \in \{1,\dots,n\} \\ \alpha_i(H) &= \gamma^{(i)} - \gamma^{(i+1)} && i = 1,\dots,n-1. \end{aligned}$$

correspond to the roots and fundamental roots respectively. Thus

$$\begin{aligned} \mathfrak{a}^+ &= \{\mathrm{diag}(\gamma^{(i)}) : \gamma^{(1)} > \gamma^{(2)} > \cdots > \gamma^{(n)},\, \gamma^{(1)} + \cdots + \gamma^{(n)} = 0\} \\ A^+ := \exp \mathfrak{a}^+ &= \{\mathrm{diag}(e^{\gamma^{(i)}}) : e^{\gamma^{(1)}} > e^{\gamma^{(2)}} > \cdots > e^{\gamma^{(n)}} \text{and determinant is } 1\} \\ A := \exp \mathfrak{a} & \end{aligned}$$

We will denote the closure of A^+ in $G = SL(n;\mathbb{C})$ by $\bar{A}^+$.

We now consider a Brownian motion on $P = SL(n;\mathbb{C})/SU(n)$. We start with a right invariant Brownian motion $\widetilde{G}$ on $SL(n;\mathbb{C})$. This means that for each $u > 0$, the process $\{\widetilde{G}_{t+u}\widetilde{G}_u^{-1} : t \geq 0\}$ is identical in law to $\widetilde{G}$ and is independent of the process $\{\widetilde{G}_r : r \leq u\}$. Thus $\widetilde{G}$ is defined multiplicatively rather than in the additive fashion of the usual Euclidean Brownian motion. This Brownian motion is obtained as a solution to the Stratonovich SDE

$$\partial\widetilde{G} = (\partial\widetilde{W})\widetilde{G} \qquad \widetilde{G}_0 \in SL(n;\mathbb{C})$$

where $\widetilde{W}$ is an additive Brownian motion on the Lie algebra $\mathfrak{sl}(n;\mathbb{C})$. By definition $\mathrm{trace}(\widetilde{W}) = 0$. Hence, we let $\tilde{\beta}$ be an $n \times n$ matrix with independent $BM(\mathbb{C})$ entries and then we take

$$\widetilde{W}_t = \tilde{\beta}_t - \frac{1}{n}\mathrm{trace}(\tilde{\beta}_t)I$$

We define

$$Y = \widetilde{G}^*\widetilde{G} \tag{3.10}$$

Y is in fact a Brownian motion on P. It has the important property that it is invariant under the mapping $\{Y_t\} \mapsto \{U_t^* Y_t U_t\}$, $U_t \in SL(n;\mathbb{C})$.

Now by the Cartan decomposition theorem for a connected semisimple Lie group (see Helgason (1978) IX Thm 1.1) we can write $G = K\bar{A}^+K$, that is, each $g \in G$ can be written as $g = k_1 a k_2$ where $k_1, k_2 \in K$ and $a \in \bar{A}^+$. Thus for any $g \in SL(n;\mathbb{C})$ we have

$$\begin{aligned} g^*g &= (k_1ak_2)^*(k_1ak_2) \\ &= k_2^*a^*ak_2 \qquad \text{since } k_1 \in SU(n) \\ &= k_2^*a^2k_2 \end{aligned} \tag{3.11}$$

We can rewrite (3.11) for the Brownian motion Y on P

$$Y_t = K(Y_t)^*[\bar{A}^+(Y_t)]^2K(Y_t) \tag{3.12}$$

and since $a \in \bar{A}^+$ implies that $a^2 \in \bar{A}^+$, this has the same form as the polar decomposition that we are interested in, as introduced in (3.8). Since Y is restricted to the regular points of P, the $a \in \bar{A}^+$ in the decomposition must be an element of A^+ and is unique. Thus, as Y_t is self-adjoint, it follows from the spectral theorem for Hermitian matrices that the matrix denoted by $[\bar{A}^+(Y_t)]^2 \in A^+$ in (3.12) is in fact the diagonal matrix with entries equal to the eigenvalues of $Y_t = \widetilde{G}_t^*\widetilde{G}_t$.

Collecting everything together we conclude that the radial part $\Gamma(Y_t)$ of our Brownian motion on the symmetric space $P = SL(n;\mathbb{C})/SU(n)$ satisfies

$$\exp\Gamma(Y_t) = \mathrm{diag}(e^{2\xi_t^{(1)}}, e^{2\xi_t^{(2)}}, \ldots, e^{2\xi_t^{(n)}}) \in A^+$$

where $\lambda_t^{(i)} := e^{2\xi_t^{(i)}}$ corresponds to the i'th largest eigenvalue of Y at time t with $\sum_{i=1}^n \xi_t^{(i)} = 0 \quad \forall t$. Now setting $\gamma_t^{(i)} = 2\xi_t^{(i)}$ for $i = 1, \ldots, n$. and rearranging we find

$$\gamma_t^{(i)} = \log\lambda_t^{(i)} \tag{3.13}$$

and thus, we have

$$\Gamma(Y_t) = \mathrm{diag}(\gamma_t^{(1)}, \gamma_t^{(2)}, \ldots, \gamma_t^{(n)}) \in \mathfrak{a}^+ \tag{3.14}$$

As previously stated, it is known that $\Gamma(Y_t)$ is in $\mathfrak{a}^+$ for all time and has generator given by the "type A" version of (3.9). We can give $\mathfrak{a}$ an Euclidean structure by equipping it with the Hilbert-Schmidt inner product which is equivalent to $1/2n$ times the Killing form in this case. Thus we identify $\mathfrak{a}$ with vectors in $\mathbb{R}^n$ having components that add up to zero. We can now state the following Corollary to Propostition 3.1.

Corollary 3.2. *Euclidean Brownian motion in $\mathfrak{a}$ (equivalently Brownian motion in $\mathbb{R}^n$ conditioned for its components to add to zero) with drift $\rho = \sum_{i<j} \epsilon_i - \epsilon_j = (n-1, n-3, \dots, 1-n)$, started at the origin and conditioned for its components to never collide, evolves like $\Gamma(Y_t)$, the process of log-transformed eigenvalues of Brownian motion on the space of positive definite Hermitian matrices of determinant 1. With appropriate initial conditions these processes are diffusions satisfying the SDE*

$$\mathrm{d}\gamma_t^{(i)} = \mathrm{d}\beta_t^{(i)} + \sum_{j \neq i} \coth(\gamma_t^{(i)} - \gamma_t^{(j)})\mathrm{d}t \tag{3.15}$$

where β is a standard Euclidean Brownian motion in $\mathbb{R}^n$ conditioned so that $\sum_{i=1}^n \beta_t^{(i)} = 0 \ \forall t$.

Note that the above SDE may be deduced from (3.3) by recalling $\coth(x) = \frac{\mathrm{d}}{\mathrm{d}x} \log(e^x - e^{-x})$.

Norris, Rogers and Williams (Norris et al. (1986)) deduce similar results for Brownian motion on the space of all positive definite symmetric matrices. They do not have a volume constraint on the ellipsoids. In other words, the restrictions on the determinants and traces are not applied as they are able to consider Brownian motion on $GL(n;\mathbb{R})/O(n)$ using a "bare hands" approach.

Similarly to the above, we may construct a Brownian motion Y^{new} on $P^{new} := GL(n;\mathbb{C})/U(n)$, which is identified with the space of all positive definite Hermitian matrices, by taking

$$Y_t^{new} := (\widetilde{G}_t^{new})^* \widetilde{G}_t^{new} \tag{3.16}$$

In this case $\widetilde{G}_t^{new}$ is a right-invariant Brownian motion on $GL(n;\mathbb{C})$, driven by the Brownian motion $\tilde{\beta}$ (as before) on the Lie algebra $\mathfrak{gl}(n;\mathbb{C})$.

Unfortunately, the space $GL(n;\mathbb{C})/U(n)$ doesn't fit directly into the framework described above. However, as noted in Taylor (1991) we can factor out the determinant, which must be real (from the Hermitian property) and positive (from the positive-definite property) and thus view $GL(n;\mathbb{C})/U(n)$ as $(SL(n;\mathbb{C})/SU(n)) \times \mathbb{R}_+$. Given this isomorphism we can decompose a Brownian motion Y_t^{new} on P^{new} in terms of a Brownian motion on $P := SL(n;\mathbb{C})/SU(n)$ and an independent multiplicative Brownian motion ($\det(Y_t^{new})^{1/n}$) on $\mathbb{R}_+$. We then see that the eigenvalues of Y_t^{new} may be written in the form

$$\lambda_t^{new} = \lambda_t \times \det(Y_t^{new})^{1/n} \tag{3.17}$$

and so take values in the space $A \times \mathbb{R}_+$. Let

$$\begin{aligned}
\gamma_t^{new} &:= \log \lambda_t^{new} \\
&= \gamma_t + \log\left(\det(Y_t^{new})^{1/n}\right) \\
&= \gamma_t + \log\Big(\big(\prod_{j=1}^{n} \lambda_t^{new\,(j)}\big)^{1/n}\Big) \\
&= \gamma_t + \frac{1}{n}\sum_{j=1}^{n} \log \lambda_t^{new\,(j)} \\
&= \gamma_t + \frac{1}{n}\sum_{j=1}^{n} \gamma_t^{new\,(j)}
\end{aligned}$$

then we have

$$\gamma_t^{new} \in \mathfrak{a} \oplus \mathbb{R}\underbrace{(1,1,1,\ldots,1,1)}_{\text{n times}} \simeq \mathbb{R}^n \tag{3.18}$$

Now observe that the fact that

$$\Big(\prod_{j=1}^{n} \lambda_t^{new\,(j)}\Big)^{1/n} = \exp\Big(\frac{1}{n}\sum_{j=1}^{n} \gamma_t^{new\,(j)}\Big)$$

is a multiplicative Brownian motion on $\mathbb{R}_+$ implies that $\frac{1}{n}\sum_{j=1}^{n} \gamma_t^{new\,(j)}$ is an additive Brownian motion in $\mathbb{R}$. It follows that the generator of γ_t^{new} is given by (c.f. (3.3))

$$\frac{1}{2}\Delta_{\mathbb{R}^n} + \left\langle \nabla \log \delta^{\frac{1}{2}}, \nabla \right\rangle$$

Thus we conclude

Corollary 3.3. *Euclidean Brownian motion in $\mathbb{R}^n$ with drift $\rho = \sum_{i<j} \epsilon_i - \epsilon_j = (n-1, n-3, \ldots, 1-n)$, started at the origin and conditioned to remain in the Weyl chamber $C_\Sigma := \{x \in \mathbb{R}^n : x_1 > x_2 > \cdots > x_n\}$, evolves like γ_t^{new}, the process of log-transformed eigenvalues of Brownian motion on the space of positive-definite Hermitian matrices. With appropriate initial conditions these processes satisfy the SDE*

$$\mathrm{d}\gamma_t^{new\,(i)} = \mathrm{d}\hat{\beta}_t^{(i)} + \sum_{j \neq i} \coth(\gamma_t^{new\,(i)} - \gamma_t^{new\,(j)})\mathrm{d}t \tag{3.19}$$

where $\hat{\beta}_t$ is a standard Euclidean Brownian motion in $\mathbb{R}^n$.

3.3. *Interpretation for types B and D.* Here we summarise how the type B and D cases are interpreted in the symmetric space setting. Further details may be found in Helgason (1978).

For the type D root system:

$$\begin{aligned}
\mathfrak{g} &= \mathfrak{so}(2n;\mathbb{C}) := \{2n \times 2n \text{ skew symmetric matrices with complex entries}\} \\
G &= SO(2n;\mathbb{C}) := \{2n \times 2n \text{ complex orthogonal matrices with det} = 1\} \\
K &= SO(2n) := \{2n \times 2n \text{ real orthogonal matrices with det} = 1\ \} \\
\mathfrak{l} &= \mathfrak{so}(2n) = \{2n \times 2n \text{ skew symmetric matrices with real entires}\}
\end{aligned}$$

We have

$$\mathfrak{so}(2n;\mathbb{C}) = \mathfrak{so}(2n) \oplus \mathfrak{p} \tag{3.20}$$

where

$$\mathfrak{p} = \{2n \times 2n \text{ skew symmetric Hermitian matrices}\}$$

Then

$$P = SO(2n;\mathbb{C})/SO(2n) \tag{3.21}$$

which may be identified with the space of $2n \times 2n$ self-adjoint, orthogonal matrices of determinant one. $\mathfrak{a}$ is the space of $2n \times 2n$ matrices with k'th 2×2 block diagonal entry

$$\begin{pmatrix} 0 & a_k \\ -a_k & 0 \end{pmatrix}$$

$a_k \in \mathbb{R}$ for $k = 1, \ldots, n$ and zeroes elsewhere. A^+ may be found by first diagonalising a^+. The roots are given by $\{\pm a_k \pm a_l\}$, the fundamental roots are equal to $\{a_i - a_{i+1}\} \cup \{a_{2n-1} + a_{2n}\}$ for $i = 1, \ldots 2n-1$ and $\rho_i = 2(2n-i)$. Again we may use the Hilbert-Schmidt inner product (which is $1/(2n-2)$ times the Killing form restricted to $\mathfrak{p}$) to give $\mathfrak{a}$ Euclidean structure.

The complex semisimple Lie algebra associated with the type B root system is $\mathfrak{so}(2n+1;\mathbb{C})$. Clearly the identities of the type D case may be adjusted accordingly to take into account the change from even to odd order. The additional 1×1 diagonal entry of $\mathfrak{a}$ is taken to be zero. The roots are the same with $\{\pm a_i\}$ as additional roots.

To obtain Brownian motion on $P = G/K$ it is sufficent to construct a Brownian motion on the Lie group G (see Taylor (1988)) . Similarly to before, this Brownian motion is obtained as the solution to the Stratonovich SDE driven by an additive Brownian motion on the Lie algebra $\mathfrak{g}$. The radial part of Brownian motion on P, equivalently Brownian motion in $\mathfrak{a}^+$ started at 0 with drift ρ, may again be identified with its eigenvalues.

4. Number variance, random matrices and an algebraic interpretation for a model of Johansson's

In what follows we will focus on the type A case of Brownian motion in $\mathbb{R}^n$ with drift $\rho := \sum_{i<j} \epsilon_i - \epsilon_j = (n-1, n-3, n-5, \ldots, 3-n, 1-n)$, started at the origin and conditioned to stay in the chamber $C_\Sigma := \{x \in \mathbb{R}^n : x_1 > x_2 > \cdots > x_n\}$. Recall from (2.14) that the law of this process at time one has density

$$\begin{aligned} & q_1^\rho(0,y) \\ = \; & \frac{h(y)}{h(\rho)} h_\rho(y)\, p_1(\rho, y) \\ = \; & \frac{1}{Z_1} \frac{\mathbb{V}_n(y)}{\mathbb{V}_n(\rho)} \prod_{1 \le i < j \le n} (1 - e^{-2(y_i - y_j)}) \prod_{1 \le i \le n} e^{-(y_i - \rho_i)^2/2} \qquad (4.1) \\ = \; & \frac{1}{Z_1} \frac{\mathbb{V}_n(y)}{\mathbb{V}_n(\rho)} \det(e^{(y_i - \rho_j)^2/2})_{i,j=1}^n \qquad (4.2) \end{aligned}$$

where $\mathbb{V}_n(x) := \prod_{1 \le i < j \le n} (x_i - x_j)$ is the Vandermonde determinant and $\rho_i = n - 2i + 1$. (4.1) is specific to the drift ρ but (4.2) is valid for more general drift vectors in C_Σ. The determinant comes from the identity $h_\rho(x) = e^{-<\rho,x>}\det(e^{x_i\rho_j})_{i,j=1}^n$ which is deduced from (2.4) given that $W(\Phi) = S_n$ in this case.

As was emphasized earlier, a consequence of the time one interchangeability of the drift and initial position is that the above density is equivalent to $q_1(\rho, y)$, the density of the law at time one of a driftless Brownian motion started from ρ and conditioned to remain in C_Σ. Such constructions of non-colliding Brownian motions with equidistant starting positions are of particular interest in the fields of nuclear and molecular physics. When considering the spectral fluctuations of a quantum system it may be necessary to assume a "mixed type" model based on an $n \times n$ matrix

$$H(\tau) := H^{\mathrm{REG}} + \tau H^{\mathrm{GUE}} \tag{4.3}$$

where H^{REG} is a fixed matrix with equispaced real eigenvalues $(u_i)_{i=1}^n$ and H^{GUE} is a matrix of the Gaussian Unitary Ensemble. If we let $\tau = \sqrt{t}$ and suppose $(u_i)_{i=1}^n$ are the starting positions for a system of n non-colliding Brownian motions, then at each time t, the distribution of the Brownian particles is equivalent to the distribution of the eigenvalues of $H(\tau)$. The transition parameter τ dictates the relative intensities of the chaotic behaviour modelled by H^{GUE} and the regular character of the harmonic oscillator type spectrum of H^{REG}. Models of this sort and some statistics related to them are discussed in Forrester (1996) and Guhr and Papenbrock (1999). Related constructions with more general starting positions corresponding to other distribution types for H^{REG} are considered in Pandey (1995); Johansson (2001).

4.1. *The number variance statistic.* The number variance, a simple statistic common in random matrix theory, may be used to gauge the changing character of such "crossover" models as the time parameter evolves or the starting positions are altered. The number variance $Var(L)$ of a point process on the real line is the variance of the number of points that fall in a typical interval of length L. Applied to the above model this translates into the variance of the number of eigenvalues of $H(\tau)$ (all real since H^{GUE} is Hermitian) in an interval or equivalently the number of Brownian particles in an interval in space at the fixed time t. If the random process of interest is spatially homogeneous then clearly it won't matter which length L interval is chosen. In other cases we might consider an appropriate average of the number variance over different intervals of the same length. We illustrate the definition with a few examples.

Example 1: A Poisson process on $\mathbb{R}$ of intensity θ
By definition, for an interval $I \subset \mathbb{R}$ we have Number$[I] \sim$ Poisson(Length$(I)\theta$) so $\mathrm{Var}(L) = \theta L$ grows linearly with interval length.

Example 2: The eigenvalues of H^{GUE}
This corresponds to the above model with $H^{\mathrm{REG}} = 0$ and $\tau = 1$. As was alluded to in the first section, the joint density of the eigenvalues is the same as the time one distribution of Dyson's Hermitian Brownian motion started from 0. It is usual to rescale the eigenvalues, based on Wigner's semicircle law, to have asymptotic mean spacing one and then to let the matrix size $n \to \infty$. The process of rescaled eigenvalues, which we shall denote $(\tilde{\lambda}_i, i \geq 1)$ is spatially homogeneous and has determinantal structure. This means that its correlation functions (also called joint

intensities) R_m have the form

$$R_m(x_1, x_2, \dots, x_m) = \det(K(x_i, x_j))_{i,j=1}^m \tag{4.4}$$

$R_m(x_1, x_2, \dots, x_m)\mathrm{d}x_1\mathrm{d}x_2 \dots \mathrm{d}x_m$ is interpreted as the probability of finding m of the points/eigenvalues in the infinitesimal intervals around $x_1, x_2, \dots x_m$. In this case the correlation kernel K is given by

$$K(x_i, x_j) = \frac{\sin \pi(x_i - x_j)}{\pi(x_i - x_j)} \tag{4.5}$$

The reader is referred to the expository account of determinantal point processes given by Soshnikov (2000) for further details.

It is well known (an explicit calculation can be found in Mehta (2004)) that the number variance of this process grows logarithmically as interval length is increased. In fact

$$Var^{\mathrm{GUE}}(L) = \frac{1}{\pi^2}(\log(2\pi L) + \gamma_{\mathrm{Euler}} + 1) + O\big(\frac{1}{L}\big) \tag{4.6}$$

Example 3: The zeroes of the Riemann zeta function.
Recall that the Riemann zeta function is given by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p \in \{\text{primes}\}} \left(1 - \frac{1}{p^z}\right)^{-1} \tag{4.7}$$

for $\mathrm{Re}(z) > 1$ and by analytic continuation for other z. Riemann famously hypothesised that all the non-trivial zeroes of $\zeta(z)$, those with non-zero imaginary part, take the form

$$z = \frac{1}{2} \pm iE_n \qquad E_n \in \mathbb{R}_+ \tag{4.8}$$

Connections between the statistical properties of the Riemann zeta function and random matrix theory are well documented (see e.g Keating and Snaith (2003) for a review). In this context, much attention has focused on the distribution of the E_n, which from now on we shall refer to as "the zeroes", on the so called critical line $Re(z) = \frac{1}{2}$. "Local" statistics of the unfolded zeroes

$$\tilde{E}_n = E_n \frac{1}{2\pi} \log \frac{E_n}{2\pi} \tag{4.9}$$

high up on the line and the corresponding statistics for the rescaled eigenvalues $\tilde{\lambda}_i$ of very large GUE matrices, actually agree. Local statistics are those such as the distribution of spacings between neighbouring points. On the other hand, when it comes to "global" statistics such as the number variance and the correlations between the nearest neighbour spacings of distant points there is a greater discrepancy.

In Berry (1988) heuristic arguments are used to propose a formula for the number variance of the Riemann zeta zeroes. He finds that for small (relative to the height E of the zeroes considered) intevals with $L < \frac{\log(E)}{2\pi}$, the number variance is in agreement with $\mathrm{Var}(L)^{\mathrm{GUE}}$ as given above. However, as interval length is increased past this level, another term in his formula gains importance, we see cancellation of terms and the number variance saturates in the sense that rather than continue to grow logarithmically, it levels off and oscillates finitely around an average value of approximately

$$\frac{1}{\pi^2}[\log\log(E/2\pi)+1.4009] \tag{4.10}$$

4.2. *A model of Kurt Johansson's.* The difference in the limiting behaviours of the number variance statistics of the second and third examples led Johansson (2004) to address the question of whether it is possible to construct a random matrix model that demonstrates number variance saturation. One of the models he considers in this quest is that of n Brownian particles started from equidistant points on the real line, with spacing a, conditioned to never collide. This corresponds to the model (4.3) with H^{REG} assigned to have eigenvalues $u_j = \Upsilon + a(n-j)$ with $\Upsilon \in \mathbb{R}$, $a \in \mathbb{R}_+$ and $j = 1, \dots, n$. Clearly our "special" starting vector $\rho := \sum_{i<j} \epsilon_i - \epsilon_j$ corresponding to the type A root system on $\mathbb{R}^n$ is a particular example of this scheme with $a = 2$ and $\Upsilon = 1-n$. The transition density for such a system of Brownian motions takes the general form

$$q_t(u,y) = \frac{1}{Z_t}\frac{\mathbb{V}_n(y)}{\mathbb{V}_n(u)}\det(e^{-(y_i-u_j)^2/2t})_{i,j=1}^n \tag{4.11}$$

which corresponds to (1.6) given $W(\Phi) = S_n$ and as expected agrees with (4.2) when $u_j = \rho_j$ and $t = 1$.

In Johansson (2001) this transition density is derived using an entirely combinatorial argument, starting with the formula of Karlin and McGregor (1959) which is equivalent to the type A version of the density (1.4). He then makes use of a theorem given in Tracy and Widom (1998) to derive the correlation kernel K_n^t of the correlation functions $R_m^{(n)}$ for the configuration of particles in space formed by the system of non-colliding Brownian motions at fixed time t . This is given by

$$K_n^t(x,y) = \frac{1}{2\pi i t}\sum_{j=1}^n e^{-(u_j-x)^2/2t}\int_{\Gamma_L} e^{(k-y)^2/2t}\prod_{i=1, i\neq j}^n \Big(\frac{k-u_i}{u_j-u_i}\Big)\mathrm{d}k \tag{4.12}$$

where $\Gamma_L : s \to L + is$. In Johansson (2004) this kernel is found to converge uniformly as $n \to \infty$ to a kernel with leading term

$$K^t_{\mathrm{approx}}(ax,ay) = \frac{1}{a}\Big(\frac{\sin\pi(x-y)}{\pi(x-y)} + \frac{d\cos\pi(x+y)+(y-x)\sin\pi(y+x)}{\pi(d^2+(y-x)^2)}\Big)$$

where

$$d = \frac{2\pi t}{a^2} \tag{4.13}$$

Note that, in contrast to the GUE case, the density of the points of this limiting determinantal process is given by

$$K^t_{\mathrm{approx}}(x,x) = \frac{1}{a}\Big(1+\frac{\cos(2\pi x/a)}{\pi d}\Big) \tag{4.14}$$

and so it is clear that the process is not spatially homogeneous. If it were, it would be of less interest here as it is known (see Soshnikov (2000)) that in such cases the number variance does not saturate. Therefore, in this case, the number variance for an interval $[b, b+L]$ will depend on b as well as on L. Johansson is able to derive an explicit formula for the number variance for such an interval by using the leading part of the kernel denoted $K^t_{\mathrm{approx}}(x,y)$ and then averaging over $b \in [0,a)$, assuming the choice is uniform. Observe that this average is equivalent to taking an average over $\Upsilon \in [0, a)$. When L is small compared to d the number variance has leading term

$$\frac{1}{\pi^2}(\log(2\pi L/a) + \gamma_{\text{Euler}} + 1) \tag{4.15}$$

which would correspond to $\text{Var}(L)^{\text{GUE}}$ had we taken a scaling limit that resulted in an asymptotic mean spacing of a (rather than 1) for the eigenvalues. If d is held constant while we let $L \to \infty$, it is found that the averaged number variance saturates to

$$\frac{1}{\pi^2}(\log(2\pi d) + \gamma_{\text{Euler}} + 1) \tag{4.16}$$

Corollary 4.1. *Consider a system of n non-colliding Brownian particles started from the "special" initial position vector ρ. The configuration of particles in space at time one, in the limit $n \to \infty$, gives rise to a determinantal process with averaged number variance saturating to the level*

$$\frac{1}{\pi^2}(\log \pi^2 + \gamma_{Euler} + 1) \tag{4.17}$$

¿From the preceding discussion this is equivalent to the number variance of the process that would be obtained as the $n \to \infty$ limit of either the configuration of particles at time one of a system of non-colliding one-dimensional Brownian motions with drift ρ all started at the origin or equivalently the eigenvalues of Y_1^{new}.

Now if we compare (4.17) and (4.15) (with $a = 2$ in the latter for a direct comparison) then it appears that adding a drift makes a significant difference as to whether or not the number variance of the limiting process saturates.

Note that since $c\rho \in C_\Sigma$ for $c \in \mathbb{R}_+$ most of the observations made in this section are valid for more general drift vectors - we just lose the symmetric space interpretation.

4.3. *A path transformation and the largest eigenvalue.* It is shown in Biane et al. (2005) that Brownian motion with drift $\mu \in C_\Sigma$ started at the origin and conditioned to remain in C_Σ can be realised by applying a certain adapted functional $\mathcal{P}_{w_0}$ to the corresponding unconditioned Brownian motion with drift, which we shall denote $(B^\mu(t), t \geq 0)$.

In the type A case, the definition of $\mathcal{P}_{w_0}$ is the same as that of the queuing theory related transformation applied to a standard driftless Brownian motion in O'Connell and Yor (2002) and to Brownian motion with drift in O'Connell (2003a). If $W(\Phi)$ is a classical Weyl group (types A, B, C and D) then, as demonstrated in Biane et al. (2005), $\mathcal{P}_{w_0}$ is equivalent to the transformation of Bougerol and Jeulin (2002) which was mentioned briefly in the last section.

w_0 is the unique element of $W(\Phi)$ of maximal length. That is, if w_0 can be written as a minimum of k reflections

$$w_0 = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k} \qquad \alpha_i \in \Sigma \tag{4.18}$$

then all other elements of $W(\Phi)$ can be written as a combination of less than k reflections. We have

$$\mathcal{P}_{w_0} = \mathcal{P}_{\alpha_1} \mathcal{P}_{\alpha_2} \dots \mathcal{P}_{\alpha_k} \tag{4.19}$$

where

$$\mathcal{P}_\alpha B^\mu(t) := B^\mu(t) - 2\inf_{s\le t}\langle \alpha, B^\mu(s)\rangle \tag{4.20}$$

In the type A case recall that $W(\Phi) \simeq S_n$. The maximal length element of S_n is written as the product of transpositions

$$w_0 = (12)(23)(34)\ldots(n-1n)(12)(23)(34)\ldots(n-2\,n-1)\ldots\ldots\ldots(12)(23)(12)$$

In this case, the first component of $\mathcal{P}_{w_0}B^\mu$ at time t is given by

$$\sup_{0=t_0<t_1<\cdots<t_n=t}\sum_{i=1}^n [B_i^\mu(t_i) - B_i^\mu(t_{i-1})].$$

The following proposition now brings together all the different elements of the paper.

Proposition 4.2. *Let $(B^\rho(t), t\ge 0)$ be a Brownian motion in $\mathbb{R}^n$ started at the origin with drift $\rho = (n-1, n-3, \ldots, 1-n)$. All the following random variables have the same distribution.*

- $\mathcal{P}_{w_0}B^\rho(1)$
- *The coordinates at time one of n-dimensional Brownian motion with drift $\rho = (n-1, n-3, \ldots, 1-n)$ started from the origin, conditioned to remain in the chamber $C_\Sigma = \{x \in \mathbb{R}^n : x_1 > x_2 > \ldots > x_n\}$.*
- *The coordinates at time one of n-dimensional driftless Brownian motion started from $\rho = (n-1, n-3, \ldots, 1-n)$, conditioned to remain in the chamber $C_\Sigma = \{x \in \mathbb{R}^n : x_1 > x_2 > \ldots > x_n\}$.*
- *The eigenvalues of Y_1^{new}, that is, the time one eigenvalues of Brownian motion on the space of all positive-definite Hermitian matrices.*
- *The eigenvalues of the ensemble (4.3) with $\tau = 1$ and $u_j = \rho_j$.*

Corollary 4.3. *In particular, the random variable*

$$\sup_{0=t_0<t_1<\cdots<t_n=1}\sum_{i=1}^n [B_i^\rho(t_i) - B_i^\rho(t_{i-1})] \tag{4.21}$$

has the same distribution as the first component/largest eigenvalue of the regimes listed in Proposition 4.2.

Corollary 4.3 should be compared with the observation of Gravner et al. (2001) and Baryshnikov (2001) that the largest eigenvalue of a GUE random matrix has the same distribution as

$$\sup_{0=t_0<t_1<\cdots<t_n=1}\sum_{i=1}^n [B_i(t_i) - B_i(t_{i-1})] \tag{4.22}$$

where $(B(t), t\ge 0)$ is a standard driftless Brownian motion in $\mathbb{R}^n$. Note that using the fact $\rho_i = n - 2i + 1$ we can re-write the expression (4.21) as

$$-n + \sup_{0=t_0<t_1<\cdots<t_n=1}\sum_{i=1}^n [t_i + B_i(t_i) - B_i(t_{i-1})]. \tag{4.23}$$

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