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Conditioned random walks and the RSK correspondence

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Abstract
We consider the stochastic evolution of three variants of the RSK algorithm, giving both analytic descriptions and probabilistic interpretations. Symmetric functions play a key role, and the probabilistic interpretations are obtained by elementary Doob–Hunt theory. In each case, the evolution of the shape of the tableau obtained via the RSK algorithm can be interpreted as a conditioned random walk. This is intuitively appealing, and can be used for example to obtain certain relationships between orthogonal polynomial ensembles. In a certain scaling limit, there is a continous version of the RSK algorithm which inherits much of the structure exhibited in the discrete settings. Intertwining relationships between conditioned and unconditioned random walks are also given. In the continuous limit, these are related to the Harish-Chandra/Itzyksen–Zuber integral.

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1. Introduction

We consider the stochastic evolution of three variants of the RSK algorithm, giving both analytic descriptions and probabilistic interpretations. Symmetric functions play a key role, and the probabilistic interpretations are obtained by the elementary Doob–Hunt theory. In each case the evolution of the shape of the tableau obtained via the RSK algorithm can be interpreted as a conditioned random walk. This is intuitively appealing, and can be used for example to obtain certain relationships between orthogonal polynomial ensembles. In a certain scaling limit, there is a continuous version of the RSK algorithm which inherits much of the structure exhibited in the discrete settings. Intertwining relationships between conditioned and unconditioned random walks are also given. In the continuous limit, these are related to the Harish-Chandra/Itzyksen–Zuber integral.

The outline of the paper is as follows. In section 2, for completeness, we recall some definitions and properties of integer partitons, tableaux, the RSK algorithms and Schur functions. In section 3, we describe the stochastic evolution of the tableaux obtained when one applies the RSK algorithms with random input. In section 4, we give probabilistic interpretations for these evolutions using the Doob–Hunt theory and, in section 5, we demonstrate the usefulness of such interpretations by deriving certain relationships between various discrete and continuous orthogonal polynomial ensembles. In section 6, we briefly describe some analogous results in a continuous setting, which were obtained earlier in [22]. These are closely connected to Hermitian Brownian motion and the GUE ensemble.

2. Some combinatorics

In this section we recall some definitions and properties of integer partitons, tableaux, the RSK algorithms and Schur functions. For more detailed accounts, see the books of Fulton [11], Stanley [27] and Macdonald [20].

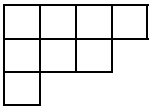
2.1. Integer partitions and tableaux

Let $\mathcal{P}$ denote the set of integer partitions

$$\left\{\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0 : |\lambda| = \sum_i \lambda_i < \infty\right\}.$$

If $|\lambda| = n$, we write $\lambda \vdash n$. The number of *parts* of λ is the number of non-zero components. It is conventional to identify partitions with the same non-zero components: for example, (4, 3, 1) and (4, 3, 1, 0, 0) are the same partition. We will also use the notation 1^n to denote the trivial partition consisting of n 1's.

The *diagram* of a partition λ is a left-justified array of boxes, λ_1 in the first row, λ_2 in the second row, and so on. For example, the diagram of the partition (4, 3, 1) is



there is a natural partial order on integer partitions, defined by $\lambda \leqslant \mu$ if the diagram of μ contains that of λ.

The *conjugate* of a partition λ, denoted by λ', is the partition whose diagram is the transpose of that of λ. For example, the conjugate of (4, 3, 1) is (3, 2, 2, 1).

A *tableau* with entries from $[k] \equiv \{1, 2, \ldots, k\}$ (the 'alphabet') is a diagram which is filled in with numbers from $[k]$ in such a way that the entries are weakly increasing from left to right along rows, and strictly increasing down the columns. The shape of a tableau T, denoted by shT, is the partition corresponding to the diagram of the tableau. For example,

1	2	2	6
3	3	4	
4			

is a tableau of shape (4, 3, 1).

Let T be a tableau with entries from $[k]$ and let α_i be the number of i's in T. The vector $\alpha \in \mathbb{N}^k$ is called the *type* of T. We denote the empty partition by ϕ. For integer partitions λ and μ, we will write $\lambda \prec \mu$ to mean that

$$\mu_1 \geqslant \lambda_1 \geqslant \mu_2 \geqslant \lambda_2 \geqslant \cdots.$$

A tableau T with entries from $[k]$ can be identified with a nested sequence of integer partitions $\phi \prec \nu^{(1)} \prec \cdots \prec \nu^{(k)}$ as follows. The partition $\nu^{(k)}$ is the shape of T, $\nu^{(k-1)}$ is the shape of the tableau obtained from T by removing all the boxes containing k, and so on. The partition $\nu^{(1)}$ has just one component equal to the number of 1's in T. The fact that $\phi \prec \nu^{(1)} \prec \cdots \prec \nu^{(k)}$ is equivalent to the fact that T is a tableau. Note that this identification gives a one-to-one correspondence between tableaux with shape λ and entries from $[k]$, and sequences

$$\phi \prec \nu^{(1)} \prec \cdots \prec \nu^{(k)} = \lambda.$$

A tableau with shape $\lambda \vdash n$ is *standard* if its entries (from $[n]$) are distinct. This is usually what is meant by a *Young tableau*, although conventions vary. What we call a tableau is sometimes referred to as a *semi-standard Young tableau*, or just *semi-standard tableau*. Recalling the correspondence between tableaux and nested partitions mentioned above, there is a correspondence between standard tableaux with shape $\lambda \vdash n$ and nested sequences $\phi = \nu^0 \nearrow \nu^{(1)} \nearrow \cdots \nearrow \nu^{(n)} = \lambda$ where $\mu \nearrow \lambda$ means $\mu \leqslant \lambda$ and $|\lambda| = |\mu| + 1$. Denote by $\mathcal{T}_m$ the set of tableaux, and by $\mathcal{S}_m$ the set of standard tableaux, with entries from $[m]$.

2.2. The Robinson–Schensted correspondence

The Robinson–Schensted correspondence is a bijective mapping from the set of 'words' $[k]^n$ to the set

$$\{(P, Q) \in \mathcal{T}_k \times \mathcal{S}_n : sh P = sh Q\}.$$

The algorithm used to define this mapping is called *row-insertion*, which takes a tableau T and an integer $i \in [k]$ and constructs a new tableau $T \leftarrow i$. The new tableau has one more box than T and has the same entries as T together with one more entry labelled i. The row-insertion procedure is performed as follows. If i is at least as large as all the entries in the first row of T, simply add a box labelled i to the end of the first row of T. Otherwise, find the leftmost entry in the first row which is strictly larger than i and remove it, relabelling that box with i. Now continue by applying the same procedure with the removed entry to the second row, and so on. Continue until a removed entry can be placed at the end of the following row, unless it has been removed from the last row, in which case it will form a new row on its own.

The Robinson–Schensted mapping is now defined as follows. Let (P, Q) denote the image of a word $a_1 \ldots a_n \in [k]^n$. Let $P^{(1)}$ be the tableau with the single entry a_1 and, for $m < n$, let $P^{(m+1)} = P^{(m)} \leftarrow a_{m+1}$. Then $P = P^{(n)}$ and Q is the standard tableau corresponding to the nested sequence $\phi \nearrow sh P^{(1)} \nearrow \cdots \nearrow sh P^{(n)}$.

An important property of the Robinson–Schensted correspondence is the fact that the length of the longest row in the tableau P is precisely the length of the longest weakly increasing subsequence in the word $a_1 \ldots a_n$.

Finally we remark that there is a variant of the Robinson–Schensted correspondence which uses *column*-insertion instead of row-insertion; the resulting mapping is again a bijection. See [11] for more details.

2.3. The Knuth generalizations

Knuth [18] generalized the Robinson–Schensted correspondence. The *RSK (Robinson–Schensted–Knuth) correspondence* is a bijective map $A \mapsto (P, Q)$ from the set of non-negative matrices $\mathbb{N}^{n\times k}$ to the set

$$\{(P, Q) \in \mathcal{T}_k \times \mathcal{T}_n : sh P = sh Q\}.$$

It has the important property that the vector of column sums of A is precisely the type of P.

This is defined as follows. Let w_m denote the word composed of A_{m1} 1's, A_{m2} 2's, and so on. The length of this word is given by the corresponding row-sum of A. For each $m \leqslant n$, let $P^{(m)}$ denote the semistandard tableau obtained when one applies the Robinson–Schensted algorithm (with *row-insertion* to the (concatenated) word $w_1 w_2 \ldots w_m$. Then set $P = P^{(n)}$ and let Q be the (semistandard) tableau corresponding to the nested sequence $\phi \prec shP^{(1)} \prec \cdots \prec shP^{(n)}$.

The above algorithm can be applied with column-insertion instead of row-insertion but the resulting Q is not a tableau (it is a reverse plane partition, see below). However, if one restricts to zero-one matrices, the Q has the property that Q^t is a tableau, and there is a correspondence: the *dual RSK correspondence* is a bijective map from zero-one matrices $\{0,1\}^{n\times k}$ to the set

$$\{(R,S) \in \mathcal{T}_k \times \mathcal{T}_n : shS = (shR)'\}.$$

2.4. Skew-tableaux

If $\mu \leqslant \lambda$, the diagrams of μ and λ are nested, and the skew-diagram λ/μ is obtained by removing all the boxes in μ from the diagram of λ. A skew-tableau with shape λ/μ is just a numbering of the boxes of λ/μ which is weakly increasing along rows and strictly increasing down columns. As before, a standard skew-tableau is one with entries which are strictly increasing along rows and columns.

As in the ordinary case, there is a one-to-one correspondence between skew-tableaux with shape λ/μ and entries from $[n]$ and nested sequences $\mu = \nu^{(0)} \prec \nu^{(1)} \prec \cdots \prec \nu^{(n)} = \lambda$. Similarly, there is a one-to-one correspondence between standard skew-tableaux with shape λ/μ and nested sequences $\mu = \nu^0 \nearrow \nu^{(1)} \nearrow \cdots \nearrow \nu^{(n)} = \lambda$.

2.5. Schur functions

Fix $k \geqslant 1$ and set $\delta = (k-1, k-2, \ldots, 0) \in \mathbb{N}^k$. Note that δ can also be identified with the integer partition $(k-1, k-2, \ldots, 1)$. For $\mu \in \mathcal{P}$ with at most k parts and a set of variables $x = (x_1, \ldots, x_k)$, set

$$a_\mu(x) = \det\left(x_i^{\mu_j}\right)_{i,j\leqslant k}. \tag{1}$$

Then a_δ is the Vandermonde function:

$$a_\delta(x) = \det\left(x_i^{k-j}\right)_{i,j\leqslant k} = \prod_{i<j\leqslant k}(x_i - x_j). \tag{2}$$

For $\lambda \in \mathcal{P}$ with at most k parts, the Schur function s_λ is defined by

$$s_\lambda = a_{\lambda+\delta}/a_\delta.$$

This is the 'classical' definition. There is also a combinatorial definition, which is equivalent and given in terms of tableaux. For $\alpha \in \mathbb{N}^k$ we write $x^\alpha = x_1^{\alpha_1}\cdots x_k^{\alpha_k}$. If T is a tableau with type α, we write $x^T = x^\alpha$. Then

$$s_\lambda(x) = \sum x^T \tag{3}$$

where the sum is over all tableaux with shape λ and entries from $[k]$. Alternatively, if we let $K_{\lambda\alpha}$ denote the number of tableau with shape λ and type α, we can write

$$s_\lambda(x) = \sum_\alpha K_{\lambda\alpha} x^\alpha. \tag{4}$$

These are the *Kostka numbers*. Note that, writing $1^n = (1, \ldots, 1)$, where the number of one's is n, $s_\lambda(1^n)$ is precisely the number of tableau with shape λ and entries from $[n]$.

The skew-Schur function $s_{\lambda/\mu}$ is also defined by (3), but in this case the sum is over all skew-tableaux with shape λ/μ and entries from $[k]$. The number of skew-tableaux with shape λ/μ and entries from $[n]$ is given by $s_{\lambda/\mu}(1^n)$. Note that this is also the number of nested sequences $\mu = \nu^{(0)} \prec \nu^{(1)} \prec \cdots \prec \nu^{(n)} = \lambda$. Schur functions have the important property that they are symmetric.

We denote the number of standard tableau with shape λ by f^{λ}, and the number of standard skew-tableaux with shape λ/μ by $f^{\lambda/\mu}$. Note that $f^{\lambda/\mu}$ is also the number of nested sequences $\mu = \nu^0 \nearrow \nu^{(1)} \nearrow \cdots \nearrow \nu^{(n)} = \lambda$.

We record the following well-known formulae for later reference:

$$f^{\lambda/\mu} = (|\lambda| - |\mu|)! \det\left[\frac{1}{(\lambda_i - \mu_j - i + j)!}\right]_{i,j\leqslant k} \tag{5}$$

and

$$s_{\lambda/\mu}(1^n) = \det\left[\binom{n-1+\lambda_i - \mu_j - i + j}{\lambda_i - \mu_j - i + j}\right]_{i,j\leqslant k}. \tag{6}$$

2.6. *Infinite tableaux and reverse plane partitions*

As we mentioned earlier, there is a one-to-one correspondence between $\mathcal{T}_n$ (tableaux with entries from $[n]$) and nested sequences of integer partitions $\{\nu^{(m)}, 0 \leqslant m \leqslant n\}$ with the property that

$$\phi = \nu^{(0)} \prec \nu^{(1)} \prec \nu^{(2)} \prec \cdots \prec \nu^{(n)}. \tag{7}$$

This also makes perfectly good sense when $n = \infty$. In this case, the correspondence is between the set $\mathcal{T}_\infty$ of (possibly infinite) tableaux with entries from the positive integers, and nested sequences $\{\nu^{(m)}, m \geqslant 0\}$ with the property that

$$\phi = \nu^{(0)} \prec \nu^{(1)} \prec \nu^{(2)} \prec \cdots. \tag{8}$$

A *reverse plane partition* is just like a tableau, except that the entries need only be weakly increasing along both rows and columns. As above, there is a one-to-one correspondence between $\mathcal{R}_n$, the set of reverse plane partitions with entries from $[n]$, and nested sequences of integer partitions

$$\phi = \pi^{(0)} \leqslant \pi^{(1)} \leqslant \pi^{(2)} \leqslant \cdots \leqslant \pi^{(n)}. \tag{9}$$

As in the tableaux setting, this correspondence extends to the infinite setting. We will denote the set of (possibly infinite) reverse plane partitions by $\mathcal{R}_\infty$.

Note that for each $1 \leqslant n \leqslant \infty$, $\mathcal{T}_n \subset \mathcal{R}_n$ but $\mathcal{T}_n \neq \mathcal{R}_n$. However, there is a one-to-one correspondence between $\mathcal{T}_\infty$ and $\mathcal{R}_\infty$: for each $T \in \mathcal{T}_\infty$ define $R \in \mathcal{R}_\infty$ by subtracting $i-1$ from all the entries in the ith row of T. In terms of nested integer partitions, this correspondence can be described as follows. Let $\{\nu^{(m)}, m \geqslant 0\}$ the corresponding sequence of integer partitions corresponding to T. Now consider the nested sequence of integer partitions $\{\pi^{(m)}, m \geqslant 0\}$ defined by

$$\pi^{(m)} = \left(\lambda_1^{(m)}, \lambda_2^{(m+1)}, \ldots\right). \tag{10}$$

Then R is the reverse plane partition corresponding to π. The 'tableau condition' (7) is precisely equivalent to the condition

$$\phi = \pi^{(0)} \leqslant \pi^{(1)} \leqslant \pi^{(2)} \leqslant \cdots. \tag{11}$$

3. The RSK algorithms with random input

3.1. The Robinson–Schensted algorithm with random words

The problem of studying the distribution of the tableaux obtained when one applies the Robinson–Schensted algorithm to a random word has been considered by Tracy and Widom [29] (see also references therein). In those papers of primary interest is the asymptotic distribution of the length of the longest weakly increasing subsequence in a random word as the length of the word goes to infinity. In [22], the 'time'-evolution of the tableaux obtained is considered as one sucessively makes insertions from an infinite random word. In this section we will recall this discussion as a preparation for presenting analagous results for the Knuth generalizations. We will denote the origin in $\mathbb{N}^k$ (and later in $\mathbb{R}^k$) by o.

Fix $k \geqslant 1$ and $w \in (0,1)^k$ with $\sum_i w_i = 1$. Let $\omega(n)$ be a sequence of independent random variables taking values in $[k]$ with common distribution

$$\mathbb{P}[\omega(n) = i] = w_i.$$

Let Z be the random walk in $\mathbb{N}^k$ defined by $Z(0) = o$ and for $n \geqslant 1, i \leqslant k$,

$$Z_i(n) = |\{m \leqslant n : \omega(m) = i\}|.$$

Let $(R^{(n)}, S^{(n)})$ be the pair of tableaux obtained when one applies the Robinson–Schensted algorithm[1] to the random word $\omega(1)\ldots\omega(n)$. Note that the type of $R^{(n)}$ is given by $Z(n)$. Using this, and the fact that the Robinson–Schensted mapping is a bijection, we have, for $(R, S) \in \mathcal{T}_k \times \mathcal{S}_n$,

$$\mathbb{P}[(R^{(n)}, S^{(n)}) = (R, S)] = w^R 1_{\{shR=shS\}}. \tag{12}$$

Set $N(0) = \phi$ and, for $n \geqslant 1$,

$$N(n) = shR^{(n)} = shS^{(n)}. \tag{13}$$

Summing (12) over pairs (R, S) with common shape λ, we obtain, for $\lambda \vdash n$,

$$\mathbb{P}[N(n) = \lambda] = s_\lambda(w) f^\lambda. \tag{14}$$

Noting that $S^{(n)}$ can be identified with the sequence $N(1), \ldots, N(n)$, we can also sum (12) over R to obtain

$$\mathbb{P}[N(1) = \lambda^{(1)}, \ldots, N(n) = \lambda^{(n)}] = s_{\lambda^{(n)}}(w) \tag{15}$$

for nested sequences $\phi \nearrow \lambda^{(1)} \nearrow \cdots \nearrow \lambda^{(n)}$.

It follows that the sequence N is a Markov chain on the set of integer partitions $\mathcal{P}$, with transition probabilities

$$\Pi_N(\mu, \lambda) = \frac{s_\lambda(w)}{s_\mu(w)} 1_{\{\mu\nearrow\lambda\}}. \tag{16}$$

In particular,

$$\sum_\lambda \Pi_N(\mu, \lambda) = \sum_{\lambda:\mu\nearrow\lambda} \frac{s_\lambda(w)}{s_\mu(w)} = 1. \tag{17}$$

Note that the law of N is symmetric in the w_i, by the symmetry of Schur functions.

[1] The arguments and conclusions that follow in this random word context are valid as stated whether one applies the algorithm with row or column insertion.

3.2. RSK with random non-negative matrices

Fix $k \geqslant 1$ and $q \in (0, 1)^k$. Let $\{\xi(m, i) : m \geqslant 1, 1 \leqslant i \leqslant k\}$ be a collection of independent random variables with

$$\mathbb{P}[\xi(m, i) = j] = (1 - q_i)q_i^j \tag{18}$$

for $j \in \mathbb{N}$. Denote by $A^{(n)}$ the $n \times k$ matrix consisting of the first n rows of the infinite matrix ξ, and by X the random walk in $\mathbb{N}^k$ defined by $X(0) = o$ and, for $n \geqslant 1$,

$$X_i(n) = \sum_{m \leqslant n} \xi(m, i). \tag{19}$$

Denote by $(P^{(n)}, Q^{(n)})$ the pair of tableau associated, by the RSK correspondence, with the random matrix $A^{(n)}$. Set $L(0) = \phi$ and, for $n \geqslant 1$,

$$L(n) = sh P^{(n)}. \tag{20}$$

Recall that the type of $P^{(n)}$ is given by $X(n)$, and the tableau $Q^{(n)}$ can be identified with the sequence

$$\phi = L(0) \prec L(1) \prec L(2) \prec \cdots \prec L(n).$$

Set $a(q) = \prod_i (1 - q_i)$.

As in the random words case, using the fact that the RSK correspondence is a bijection, and the fact that $X(n)$ is the type of $P^{(n)}$, we have, for $(P, Q) \in \mathcal{T}_k \times \mathcal{T}_n$,

$$\mathbb{P}[(P^{(n)}, Q^{(n)}) = (P, Q)] = a(q)^n q^P 1_{\{shP=shQ\}}. \tag{21}$$

Summing (21) over (P, Q) with $shP = shQ = \lambda$ we obtain

$$\mathbb{P}[L(n) = \lambda] = a(q)^n s_\lambda(q) s_\lambda(1^n). \tag{22}$$

Summing over P we obtain, if $shQ = \lambda$,

$$\mathbb{P}[Q^{(n)} = Q] = a(q)^n s_\lambda(q). \tag{23}$$

Thus, for any nested sequence

$$\phi = \lambda^{(0)} \prec \lambda^{(1)} \prec \lambda^{(2)} \prec \cdots \prec \lambda^{(n)} \tag{24}$$

we have

$$\mathbb{P}[L(1) = \lambda^{(1)}, \ldots, L(n) = \lambda^{(n)}] = a(q)^n s_{\lambda^{(n)}}(q). \tag{25}$$

This implies that L is a Markov chain on $\mathcal{P}$ with transition probabilities

$$\Pi_L(\mu, \lambda) = a(q) \frac{s_\lambda(q)}{s_\mu(q)} 1_{\{\lambda \succ \mu\}}. \tag{26}$$

We remark here that, in particular,

$$\sum_\lambda \Pi_L(\mu, \lambda) = \sum_{\lambda \succ \mu} a(q) \frac{s_\lambda(q)}{s_\mu(q)} = 1. \tag{27}$$

This is sometimes called *Pieri's formula*, although there are many related identities in the literature which are given this title. Since the Schur functions are symmetric, the law of L is symmetric in the q_i.

3.3. Dual RSK with random zero-one matrices

Fix $k \geqslant 1$ and $p \in (0,1)^k$. Let $\{\eta(m,i) : m \geqslant 1, 1 \leqslant i \leqslant k\}$ be a collection of independent random variables taking values in $\{0,1\}$ with

$$\mathbb{P}[\eta(m,i) = 1] = p_i = 1 - \mathbb{P}[\eta(m,i) = 0] \tag{28}$$

for all m. Denote by $B^{(n)}$ the $n \times k$ matrix consisting of the first n rows of the infinite matrix η, and let Y be the random walk in $\mathbb{N}^k$ with $Y(0) = o$ and, for $n \geqslant 1$,

$$Y_i(n) = \sum_{m \leqslant n} \eta(m,i).$$

Denote by $(T^{(n)}, U^{(n)})$ the pair of tableaux associated, by the dual RSK correspondence, with the random matrix $B^{(n)}$; set $M(0) = \phi$ and, for $n \geqslant 1$,

$$M(n) = shT^{(n)} = (shU^{(n)})'. \tag{29}$$

Note that the type of $T^{(n)}$ is given by $Y(n)$.

The sequence $U^{(n)}$ is nested, in the sense that $U^{(n-1)}$ can be obtained from $U^{(n)}$ by removing all the boxes in $U^{(n)}$ containing the number n. The reverse plane partition $(U^{(n)})^t$ can be identified with the nested sequence

$$\phi = M(0) \leqslant M(1) \leqslant M(2) \leqslant \cdots \leqslant M(n). \tag{30}$$

Since the dual RSK correspondence is a bijection, and $Y(n)$ is the type of $T^{(n)}$, we have, for $(T,U) \in \mathcal{T}_k \times \mathcal{T}_n$,

$$\mathbb{P}[(T^{(n)}, U^{(n)}) = (T,U)] = a(p)^n \rho^T 1_{\{shT=(shU)'\}} \tag{31}$$

where $\rho = (\rho_1, \ldots, \rho_k)$, $\rho_i = (1-p_i)^{-1} p_i$.

Summing over (T,U) with $shT = (shU)' = \lambda$ we obtain

$$\mathbb{P}[M(n) = \lambda] = a(p)^n s_\lambda(\rho) s_{\lambda'}(1^n). \tag{32}$$

Summing over P we obtain, if $shU = \lambda'$ and $\lambda \leqslant n.1^k$,

$$\mathbb{P}[U^{(n)} = U] = a(p)^n s_\lambda(\rho). \tag{33}$$

Thus, for any nested sequence

$$\phi = \lambda^{(0)} \leqslant \lambda^{(1)} \leqslant \lambda^{(2)} \leqslant \cdots \leqslant \lambda^{(n)} \tag{34}$$

with $\lambda^{(m)} \in \lambda^{(m-1)} + \{0,1\}^k$, for each $m \leqslant n$,

$$\mathbb{P}[M(1) = \lambda^{(1)}, \ldots, M(n) = \lambda^{(n)}] = a(p)^n s_{\lambda^{(n)}}(\rho). \tag{35}$$

This implies that M is a Markov chain on $\mathcal{P}$ with transition probabilities

$$\Pi_M(\mu,\lambda) = a(p) \frac{s_\lambda(\rho)}{s_\mu(\rho)} 1_{\{\lambda \in \mu + \{0,1\}^k\}}. \tag{36}$$

In particular,

$$\sum_\lambda \Pi_M(\mu,\lambda) = \sum_{\lambda \in \mu + \{0,1\}^k} a(p) \frac{s_\lambda(\rho)}{s_\mu(\rho)} = 1. \tag{37}$$

Again, since the Schur functions are symmetric, the law of M is symmetric in the p_i.

3.4. *Connection with Schur measures and $\beta = 2$ ensembles*

The probability measures on partitions defined by (14) and (22) are particular specializations of *Schur measures*, and the fact that they appear in this context is well-known (see, for example [7, 14]). Similarly, the Markov chains L and M are particular specializations of the *Schur process* discussed in [24].

When all the components of w are equal, the probability measure on partitions defined by (14) is the *de-Poissonized Charlier ensemble*:

$$\mathbb{D}_n^{(k)}[\lambda] = s_\lambda(1^k/k) f^\lambda = k^{-|\lambda|} s_\lambda(1^k) f^\lambda 1_{\{\lambda\vdash n\}}. \tag{38}$$

The *Charlier ensemble* is given by

$$\mathbb{C}^{(k)}[\lambda] = \sum_{n\geqslant 0} \mathrm{e}^{-k}\frac{k^n}{n!}\mathbb{D}_n^{(k)}[\lambda] \tag{39}$$

$$= \mathrm{e}^{-k}\frac{1}{|\lambda|!} s_\lambda(1^k) f^\lambda \tag{40}$$

$$= Z^{-1}\prod_{i<j}(\lambda_i - \lambda_j + i - j)^2 \prod_i \frac{1}{\lambda_i!} \tag{41}$$

where Z is a normalization constant.

When all the components of q are equal, $q_i = \theta$ say, the probability measure on partitions defined by (22) is the *Meixner ensemble*:

$$\mathbb{M}_{\theta,n}^{(k)}[\lambda] = (1-\theta)^{kn}\theta^{|\lambda|} s_\lambda(1^k) s_\lambda(1^n). \tag{42}$$

The Charlier ensemble can be regarded as a limiting case of the Meixner ensemble: as $n \to \infty$, the sequence $\mathbb{M}_{1/n,n}^{(k)}$ converges weakly to $\mathbb{C}^{(k)}$.

The *Laguerre ensemble*, a continuous distribution on

$$\{x \in \mathbb{R}^k : x_1 \geqslant x_2 \geqslant \cdots \geqslant x_k \geqslant 0\}$$

defined by

$$\mathbb{L}_n^{(k)}[\mathrm{d}x] = Z^{-1}\prod_{i<j}(x_i - x_j)^2 \prod_i x_i^{n-k}\,\mathrm{e}^{-x_i}\,\mathrm{d}x \tag{43}$$

can also be recovered from the Meixner ensemble by rescaling in the limit as $\theta \to 1$. More precisely, if $L^{(N)}$ is distributed according to the measure $\mathbb{M}_{1-N^{-1},n}^{(k)}$ then, as $N \to \infty$, the law of $N^{-1}L^{(N)}$ converges weakly to $\mathbb{L}_n^{(k)}$.

When all the components of p are equal, $p_i = \tau$ say, the probability measure on partitions defined by (32) is the *Krawchouk ensemble*:

$$\mathbb{K}_{\tau,n}^{(k)}[\lambda] = \tau^{|\lambda|}(1-\tau)^{kn-|\lambda|} s_\lambda(1^k) s_{\lambda'}(1^n). \tag{44}$$

The Charlier ensemble can also be seen as limiting case of the Krawchouk ensemble: as $n \to \infty$, the sequence $\mathbb{K}_{1/n,n}^{(k)}$ converges weakly to $\mathbb{C}^{(k)}$.

4. Interpretations

4.1. *A theorem of Doob*

In this section, for later reference, we recall some elementary potential theory for Markov chains. For more extensive accounts, see [30] or [10]. Let Σ be a countably infinite set and let $\Pi : \Sigma \times \Sigma \to [0, 1]$ be a substochastic transition matrix, that is

$$\sum_y \Pi(x, y) \leqslant 1$$

for all $x \in \Sigma$. The Green's function associated with Π is defined by

$$\Gamma(x, y) = \sum_{n \geqslant 0} \Pi^n(x, y).$$

We will assume that there exists $x^* \in \Sigma$ with the property that, for each $y \in \Sigma, 0 < \Gamma(x^*, y) < \infty$.

A function $h : \Sigma \to \mathbb{R}$ is harmonic for Π if $\Pi h = h$, that is

$$\sum_y \Pi(x, y) h(y) = h(x)$$

for all $x \in \Sigma$. Consider a strictly positive function h which is harmonic for Π. Define a new matrix

$$\Pi_h(x, y) = \frac{h(y)}{h(x)} \Pi(x, y)$$

and note that, since h is harmonic for Π, this is a stochastic transition matrix, that is

$$\sum_y \Pi_h(x, y) = 1$$

for all $x \in \Sigma$. This is called the Doob h-transform of Π. We can now consider the corresponding Markov chain X, with $X(0) = x^*$ and transition probabilities given by

$$\mathbb{P}[X(n+1) = y | X(n) = x] = \Pi_h(x, y).$$

The following theorem is due to Doob.

Theorem 4.1. *Suppose that, for some fixed $f \in \mathbb{R}^\Sigma$,*

$$\lim_{n \to \infty} \frac{\Gamma(\cdot, X(n))}{\Gamma(x^*, X(n))} = f$$

(pointwise) almost surely. Then $h = Cf$ for some constant C.

4.2. A useful concentration inequality

Let W be a random walk in $\mathbb{N}^k$ with $W(0) = o, \mathbb{E}W(1) = \vartheta \in \mathbb{R}_+^k$, and assume that $\mathbb{E}\exp(\rho \cdot W(1))$ exists and is finite for ρ in a neighbourhood of the origin. Then

$$\mathbb{P}[|W(n) - \vartheta n| > \epsilon n] \leqslant K \mathrm{e}^{-c(\epsilon) n}, \tag{45}$$

where K and c are constants which do not depend on n.

4.3. Random words example

First we note that Z is a Markov chain on $\mathbb{N}^k$ with transition matrix

$$\Pi_Z(\alpha, \beta) = w^{\beta - \alpha} 1_{\{\alpha \nearrow \beta\}}. \tag{46}$$

It is convenient to identify the set of integer partitions, with at most k parts, with the set

$$\Omega = \{\alpha \in \mathbb{N}^k : \alpha_1 \geqslant \cdots \geqslant \alpha_k\}.$$

Let $\Pi_{Z,\Omega}$ be the restriction of Π_Z to Ω, and note that this is a sub-stochastic transition matrix on Ω. Now we can write

$$\Pi_N(\mu, \lambda) = \frac{f(\lambda)}{f(\mu)} \Pi_{Z,\Omega}(\mu, \lambda), \tag{47}$$

where $f(\lambda) = w^{-\lambda} s_\lambda(w)$. The fact that Π_N is a transition matrix is equivalent to the fact that f is a positive harmonic function for $\Pi_{Z,\Omega}$ and we see that Π_N is the corresponding Doob transform of $\Pi_{Z,\Omega}$.

Denote the law of Z started from $Z(0) = \lambda$ by $\mathbb{P}_\lambda$.

Lemma 4.2. *Suppose $w_1 > \cdots > w_k$. Then, for all $\lambda \in \Omega$,*

$$\psi(\lambda) := \mathbb{P}_\lambda[Z(n) \in \Omega, n \geqslant 0] > 0. \tag{48}$$

Proof. This is easily verified using the concentration inequality (45). □

Thus, if $w_1 > \cdots > w_k$, we can consider the chain Z conditioned never to leave Ω. This is a Markov chain on Ω with transition matrix

$$\hat{\Pi}_{Z,\Omega}(\mu, \lambda) = \frac{\psi(\lambda)}{\psi(\mu)} \Pi_{Z,\Omega}(\mu, \lambda). \tag{49}$$

Note that this is also a Doob transform of $\Pi_{Z,\Omega}$.

Theorem 4.3. *Suppose $w_1 > \cdots > w_k$. Then $\psi(\lambda) = w^{-\lambda-\delta} a_{\lambda+\delta}(w)$ and $\Pi_N = \hat{\Pi}_{Z,\Omega}$. That is, N has the same law as that of Z conditioned never to leave Ω.*

Proof. Let $\mu \leqslant \lambda$ with $|\lambda| - |\mu| = n$ and recall that the number of 'paths'

$$\mu = \lambda^{(0)} \nearrow \lambda^{(1)} \nearrow \cdots \nearrow \lambda^{(n)} = \lambda \tag{50}$$

is the same as the number of standard skew-tableaux with shape λ/μ, and this is given by $f^{\lambda/\mu}$. Thus,

$$\Pi^n_{Z,\Omega}(\mu, \lambda) = \mathbb{P}_\mu[Z(n) = \lambda;\ Z(m) \in \Omega, m \leqslant n] \tag{51}$$

$$= w^{\lambda-\mu} f^{\lambda/\mu} 1_{\{\lambda \geqslant \mu, |\lambda|-|\mu|=n\}} \tag{52}$$

and the corresponding Green function is given by

$$\Gamma_{Z,\Omega}(\mu, \lambda) = \sum_{n \geqslant 0} \Pi^n_{Z,\Omega}(\mu, \lambda) = w^{\lambda-\mu} f^{\lambda/\mu}. \tag{53}$$

If $\lambda \to \infty$ in the direction w, then

$$\frac{\Gamma_{Z,\Omega}(\mu, \lambda)}{\Gamma_{Z,\Omega}(\phi, \lambda)} \to w^{-\mu} s_\mu(w). \tag{54}$$

This is easily verified using the formula (5).

On the other hand, if $\hat{\mathbb{P}}$ denotes the law of the conditioned walk (with transition matrix $\hat{\Pi}_{Z,\Omega}$) started at ϕ, then, using (45),

$$\hat{\mathbb{P}}[|Z(n) - wn| > \epsilon n] \tag{55}$$

$$= \psi(\phi)^{-1} \mathbb{P}[|Z(n) - wn| > \epsilon n;\ Z(m) \in \Omega, \forall m \geqslant 0] \tag{56}$$

$$\leqslant \psi(\phi)^{-1} \mathbb{P}[|Z(n) - wn| > \epsilon n] \tag{57}$$

$$\leqslant \psi(\phi)^{-1} K \mathrm{e}^{-c(\epsilon)n}. \tag{58}$$

It follows that $Z(n)/n \to w$ almost surely with respect to $\hat{\mathbb{P}}$. Thus, by theorem 4.1, we must have $\psi = Cf$, for some constant $C = C(w)$, and $\Pi_N = \hat{\Pi}_{Z,\Omega}$.

All that remains is to identify the constant C. To do this, first we note that, if $\lambda \to \infty$ in the direction w, then $\psi(\lambda) \to 1$. This can be verified using the concentration inequality. On the other hand, we have $f(\lambda) \to w^{\delta} a_{\delta}(w)^{-1}$. It follows that $C(w) = w^{-\delta} a_{\delta}(w)$ and so

$$\psi(\lambda) = w^{-\delta} a_{\delta}(w) w^{-\lambda} s_{\lambda}(w) = w^{-\lambda-\delta} a_{\lambda+\delta}(w)$$

as required. □

The following intertwining relationship also holds.

Theorem 4.4.

$$\Pi_N \Lambda_w = \Lambda_w \Pi_Z \tag{59}$$

where

$$\Lambda_w(\lambda, \alpha) = \mathbb{P}[Z(n) = \alpha | N(m), m \leqslant n; N(n) = \lambda] = \frac{w^{\alpha} K_{\lambda\alpha}}{s_{\lambda}(w)}. \tag{60}$$

Proof. Here is a sketch; see [22] for more details. The identity

$$\mathbb{P}[Z(n) = \alpha | N(m), m \leqslant n; N(n) = \lambda] = \frac{w^{\alpha} K_{\lambda\alpha}}{s_{\lambda}(w)}$$

follows immediately from (12) and the combinatorial definition of the Schur functions. The intertwining relationship $\Pi_N \Lambda_w = \Lambda_w \Pi_Z$ is equivalent to the statement that, for all $q \in \mathbb{R}_+^k$,

$$\sum_{\beta} q^{\beta} \sum_{\lambda} \Pi_N(\mu, \lambda) \Lambda_w(\lambda, \beta) = \sum_{\beta} q^{\beta} \sum_{\alpha} \Lambda_w(\mu, \alpha) \Pi_Z(\alpha, \beta)$$

and this is readily verified using the identity (17). □

Remarks.

1. Note that, if the w_i's are all equal,

   $$\Pi^n_{Z,\Omega}(\phi, \lambda) = k^{-n} f^{\lambda} 1_{\lambda \vdash n}.$$

   Normalized, this is a $\beta = 1$ version of the Plancherel measure; Poissonized, it is a $\beta = 1$ version of the Charlier ensemble. The interpretation of the normalized measure is the following: it is the conditional distribution of $Z(n)$, given that $Z(m) \in \Omega$ for all $m \leqslant n$. Note that the partition function $\sum_{\lambda \vdash n} f^{\lambda}$ is just the number of involutions in the permutation group S_n. This observation is consistent with recent work by Katori and Tanemura on viscious walkers [16], where a similar interpretation is given for the GOE.
2. Some of the above results were presented in [19] and [22]. A more extensive treatment on the Martin boundary of the Young lattice, a closely related problem, was initiated by Thoma [28], and later developed by Vershik and Kerov and others (see, for example [17]). See also [9].
3. Given the recent connections made in [22], theorem 4.3 is equivalent to the representation theorem for non-colliding random walks obtained in [23], which was obtained using reversibility and symmetry properties of queues in series. In the case $k = 2$, it is equivalent to a discrete version of Pitman's $2M - X$ theorem.
4. Most of the above discussion makes sense when $k = \infty$. The probabilistic interpretation given in theorem 4.3 requires that the probability $\psi(w) > 0$, which imposes a certain restriction on the rate that w_i goes to zero as $i \to \infty$.
5. For more on intertwinings and related work on quantum random walks, see [3–6].

4.4. RSK example

First we note that X is a Markov chain with state space $\mathbb{N}^k$ and transition probabilities

$$\Pi_X(\alpha, \beta) = a(q)q^{\beta-\alpha}1_{\{\beta \geqslant \alpha\}}. \tag{61}$$

Consider the substochastic transition matrix on Ω defined by

$$\Pi_{X,\prec}(\mu, \lambda) = \Pi_X(\mu, \lambda)1_{\{\lambda \succ \mu\}}. \tag{62}$$

Then we can write

$$\Pi_L(\mu, \lambda) = \frac{g(\lambda)}{g(\mu)}\Pi_{X,\prec}(\mu, \lambda) \tag{63}$$

where $g(\lambda) = q^{-\lambda}s_\lambda(q)$. The fact that Π_L is a transition matrix is equivalent to the fact that g is a positive harmonic function for $\Pi_{X,\prec}$ and we see that Π_L is the corresponding Doob transform of $\Pi_{X,\prec}$.

Let $\mathbb{P}_\lambda$ denotes the law of the chain X started at $X(0) = \lambda$, and consider the event

$$E = \{X(0) \prec X(1) \prec X(2) \prec \cdots\}.$$

Lemma 4.5. *Suppose $q_1 > \cdots > q_k$. Then, for all $\lambda \in \Omega$,*

$$\varphi(\lambda) := \mathbb{P}_\lambda(E) > 0.$$

Proof. As in the random words case, this is easily verifed using the concentration inequality (45). □

We can therefore consider the sequence X conditioned on the event E. This is a Markov chain on Ω with transition matrix

$$\hat{\Pi}_{X,\prec}(\mu, \lambda) = \frac{\varphi(\lambda)}{\varphi(\mu)}\Pi_{X,\prec}(\mu, \lambda). \tag{64}$$

Theorem 4.6. *Suppose $q_1 > \cdots > q_k$. Then $\mathbb{P}_\lambda(E) = q^{-\lambda-\delta}a_{\lambda+\delta}(q)$ and $\Pi_L = \hat{\Pi}_{X,\prec}$. That is, the sequence L has the same distribution as that of X conditioned on the event E.*

Proof. Let $\mu \leqslant \lambda$ and recall that the number of paths

$$\mu = \lambda^{(0)} \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(n)} = \lambda \tag{65}$$

is precisely the number of skew-tableau with shape λ/μ and entries from $[n]$, which is given by $s_{\lambda/\mu}(1^n)$. Thus,

$$\Pi^n_{X,\prec}(\mu, \lambda) = \mathbb{P}_\mu[X(n) = \lambda;\, X(m-1) \prec X(m), m \leqslant n] \tag{66}$$

$$= a(q)^n q^{\lambda-\mu} s_{\lambda/\mu}(1^n) \tag{67}$$

and the corresponding Green function is given by

$$\Gamma_{X,\prec}(\mu, \lambda) = \sum_{n \geqslant 0} \Pi^n_{Z,\prec}(\mu, \lambda) \tag{68}$$

$$= q^{\lambda-\mu} \sum_{n \geqslant 0} a(q)^n s_{\lambda/\mu}(1^n). \tag{69}$$

If $\lambda \to \infty$ in the direction γ, then

$$\frac{\Gamma_{Z,\prec}(\mu, \lambda)}{\Gamma_{Z,\prec}(\phi, \lambda)} \to q^{-\mu}s_\mu(q). \tag{70}$$

This can be verified using the formula (6). It is not as straightforward as in the random words case, as it requires a saddle point analysis of the summation at $n = |\gamma|^{-1}|\lambda|$. We omit the details.

On the other hand, if $\hat{\mathbb{P}}$ denotes the law of the conditioned walk (with transition matrix $\hat{\Pi}_{X,\prec}$) started at ϕ, then

$$\hat{\mathbb{P}}[|X(n) - \gamma n| > \epsilon n] \leqslant \varphi(\phi)^{-1}\mathbb{P}[|X(n) - \gamma n| > \epsilon n] \tag{71}$$

$$\leqslant \varphi(\phi)^{-1} K \mathrm{e}^{-c(\epsilon)n}, \tag{72}$$

which implies that $X(n)/n \to \gamma$ almost surely with respect to $\hat{\mathbb{P}}$.

By theorem 4.1, it follows that $\varphi(\mu) = Cf(\mu)$, for some constant $C = C(q)$. Again, using the concentration inequality (45) we can show that, if $\lambda \to \infty$ in the direction γ, then $\varphi(\lambda) \to 1$, and also that $g(\lambda) \to q^{\delta} a_{\delta}(q)^{-1}$. □

As in the random words case, we have the following intertwining relationship. We omit the details, as they are essentially the same as in the random words case. The intertwining relationship in this case is verified using the Pieri formula (27).

Theorem 4.7.

$$\Pi_L \Lambda_q = \Lambda_q \Pi_X \tag{73}$$

where

$$\Lambda_q(\lambda, \alpha) = \mathbb{P}[X(n) = \alpha | L(m), m \leqslant n; L(n) = \lambda] = \frac{q^{\alpha} K_{\lambda\alpha}}{s_{\lambda}(q)}. \tag{74}$$

The random sequence L can be thought of as an infinite random tableau, so it natural to consider the corresponding random reverse plane partition. This is given by the nested sequence $\tilde{L}$, where

$$\tilde{L}(n) = (L_1(n), \ldots, L_k(n + k - 1)). \tag{75}$$

Corollary 4.8. *The sequence $\tilde{L}$ has the same law as that of X conditioned never to exit Ω.*

Proof. Set

$$\tilde{X}(n) = (X_1(n), \ldots, X_k(n + k - 1)) \tag{76}$$

and note that

$$E = \{X(0) \prec X(1) \prec X(2) \prec \cdots\} \tag{77}$$

$$= \{\tilde{X}(n) \in \Omega, \forall n \geqslant 0\} \tag{78}$$

$$= \{\tilde{X}(n) \in \Omega, \forall n \geqslant 0; \tilde{X}(0) = \phi\}. \tag{79}$$

Note also that the law of X is the same as the conditional law of $\tilde{X}$ given $\tilde{X}(0) = \phi$. This completes the proof. □

Again, we remark that, if the parameters q_i are all equal, the normalized measure $\Pi^n(\phi, \lambda)/Z$ is a $\beta = 1$ version of the Meixner ensemble.

4.5. Dual RSK example

The sequence Y is a Markov chain with transition probabilities

$$\Pi_Y(\alpha, \beta) = a(p)\rho^{\beta-\alpha} 1_{\{\beta\in\alpha+\{0,1\}^k\}}. \tag{80}$$

We denote the restriction to Ω by $\Pi_{Y,\Omega}$ and note that Π_M is the Doob transform of $\Pi_{Y,\Omega}$ via the harmonic function $h(\lambda) = \rho^{-\lambda} s_\lambda(\rho)$.

Denote the law of Y started at μ by $\mathbb{P}_\mu$. Using the concentration inequality (45) we can show that

Lemma 4.9. *Suppose $p_1 > \cdots > p_k$. Then, for any $\lambda \in \Omega$,*

$$\kappa(\lambda) := \mathbb{P}_\lambda[Y(n) \in \Omega, n \geqslant 0] > 0. \tag{81}$$

We can therefore condition Y never to exit Ω. The conditioned walk has transition matrix

$$\hat{\Pi}_{Y,\Omega}(\mu, \lambda) = \frac{\kappa(\lambda)}{\kappa(\mu)} \Pi_{Y,\Omega}(\mu, \lambda) \tag{82}$$

which we see is the Doob transform of $\Pi_{Y,\Omega}$ via the harmonic function κ.

Theorem 4.10. *Suppose $p_1 > \cdots > p_k$. Then $\kappa(\lambda) = \rho^{-\lambda-\delta} a_{\lambda+\delta}(\rho)$ and $\Pi_M = \hat{\Pi}_{Y,\Omega}$. That is, the sequence M has the same law as that of Y conditioned never to exit Ω.*

Proof. The proof is the same as in the other cases. The Green function in this case is given by

$$\Gamma_{Y,\Omega}(\mu, \lambda) = \rho^{\lambda-\mu} \sum_{n\geqslant 0} a(p)^n s_{\lambda'/\mu'}(\rho) \tag{83}$$

and computing the asymptotics again requires some saddle point analysis. □

We also have the intertwining relationship:

Theorem 4.11.

$$\Pi_M \Lambda_\rho = \Lambda_\rho \Pi_Y$$

where

$$\Lambda_\rho(\lambda, \alpha) = \mathbb{P}[Y(n) = \alpha | M(m), m \leqslant n; M(n) = \lambda] = \frac{\rho^\alpha K_{\lambda\alpha}}{s_\lambda(\rho)}. \tag{84}$$

The connection given in [22] between the Robinson–Schensted algorithm, with column insertion, and the path-transformation of [23], carries over the context of the dual RSK correspondence; given this, theorem 4.10 above is equivalent to the representation for the ‘Krawchouk process’ presented in [19].

5. Inter-relationships

Let x be a weakly increasing sequence in $\mathbb{N}^k$ with $x(0) = \phi$. Define $y = \mathcal{I}(x)$, by

$$y_i(m) = \min\{n \geqslant 0 : x_{k-i+1}(n) \geqslant m\}. \tag{85}$$

Note that $\mathcal{I}(y) = x$, and $y_i(m) \leqslant n$ if, and only if, $x_{k-i+1}(n) \geqslant m$. Note also that $x(n) \in \Omega$ for all $n \geqslant 0$ if, and only if, $y(n) \in \Omega$ for all $n \geqslant 0$.

We will write $X^{(q)}$, $\tilde{L}^{(q)}$ and so on, to express the dependence of these processes on the parameter q. Let $p_i = 1 - q_{k-i+1}$, $i \leqslant k$.

Theorem 5.1. *$\mathcal{I}(\tilde{L}^{(q)})$ has the same law as $\tilde{L}^{(p)}$.*

Proof. This follows from corollary 4.8 and the fact that $\mathcal{I}(X^{(q)})$ has the same law as $X^{(p)}$. □

Let $\mathrm{d}(n) = (n, \ldots, n) \in \mathbb{N}^k \qquad n \geqslant 0$.

Theorem 5.2. *$\mathcal{I}(d + \tilde{L}^{(q)})$ has the same law as $M^{(p)}$.*

Proof. This follows from theorem 4.10, corollary 4.8 and the fact that $\mathcal{I}(d + X^{(q)})$ has the same law as $Y^{(p)}$. □

Corollary 5.3.

$$\sum_{\lambda_1 \geqslant m} \mathbb{M}^{(k)}_{\theta,n}[\lambda] = \sum_{\lambda_k \leqslant n} \mathbb{M}^{(k)}_{1-\theta,m+k-1}[\lambda] \tag{86}$$

$$\sum_{\lambda_1 \geqslant m-n} \mathbb{M}^{(k)}_{\theta,n}[\lambda] = \sum_{\lambda_k \leqslant n} \mathbb{K}^{(k)}_{1-\theta,m}[\lambda] \tag{87}$$

$$\sum_{\lambda_k \geqslant m-n-k+1} \mathbb{M}^{(k)}_{\theta,n+k-1}[\lambda] = \sum_{\lambda_1 \leqslant n} \mathbb{K}^{(k)}_{1-\theta,m}[\lambda] \tag{88}$$

We remark that the identity (88) appears as lemma 2.9 in [15] and is also discussed in [19].

Taking limits in (86) we recover the following relationship between the Laguerre and Charlier ensembles.

Corollary 5.4.

$$\sum_{\lambda_1 \geqslant m} \mathbb{C}^{(k)}[\lambda] = \int_{x_k < 1} \mathbb{L}^{(k)}_{m+k-1}[\mathrm{d}x]. \tag{89}$$

The distribution of $\tilde{L}(n)$ seems 'complicated, but not uninteresting'[2]. It is possible to show that $|\tilde{L}|$ had the same law as $|X|$.

6. Conditioned Brownian motion and RSK for continuous functions

This is a summary of some results which were presented in [22]. It is not immediately clear how one would extend the notions of 'words' and 'tableaux' to a continuous setting. To motivate what follows, we will first make some identifications in the discrete setting which one can expect to have continuous analogues.

A word $a_1 \ldots a_n \in [k]^n$ can be identified with a 'path' $x(1), \ldots, x(n)$ in $\mathbb{N}^k$ by setting

$$x_i(m) = |\{l \leqslant m : a_l = i\}|.$$

Let $C_{k,n}$ denote the image of $[k]^n$ under this identification.

As we discussed in section 2, a tableau P, with shape $\lambda \vdash n$ and entries from $[k]$, can be identified with a nested sequence $\phi \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(k)} = \lambda$. Denote the set of such sequences by $N_{k,\lambda}$.

Similarly, the set of standard tableaux with shape $\lambda \vdash n$ can be identified with the set of 'paths' $\phi \nearrow \nu^{(1)} \nearrow \cdots \nearrow \nu^{(n)} = \lambda$; denote this set of paths by $R_{k,n,\lambda}$ and note that $R_{k,n,\lambda} \subset C_{k,n}$.

[2] As statistician Jerzy Neymann once famously described life.

We can therefore regard the Robinson–Schensted mapping as a bijection between the sets $C_{k,n}$ and

$$\bigcup_{\lambda\vdash n} N_{k,\lambda} \times R_{k,n,\lambda}.$$

It was shown in [22] that this definition extends to a continuous setting when k is fixed and n is replaced by a continuous variable $t > 0$. In this setting, a 'word of length t' is a continuous function $f : [0, t] \to \mathbb{R}^k$ with $f(0) = o$. We denote the set of such words by $\mathcal{C}_{k,t}$.

Set

$$W_i = \{x \in \mathbb{R}^i : x_1 \geqslant \cdots \geqslant x_i\}.$$

The continuous analogue of a semistandard tableau with entries from $[k]$ is a nested sequence

$$y^{(1)} \prec \cdots \prec y^{(k)} = y$$

where $y^{(i)} \in \mathbb{R}^i$, for each $i \leqslant k$, and we extend the definition of the relation $\prec$ to: $x \prec y$ if $x \in \mathbb{R}^i$ and $y \in \mathbb{R}^{i+1}$ for some i, and

$$y_1 \geqslant x_1 \geqslant y_2 \geqslant x_2 \geqslant \cdots y_i \geqslant x_i \geqslant y_{i+1}.$$

Denote the set of such sequences by $\mathcal{N}_{k,y}$.

The continuous analogue of a standard tableau, 'of shape $y \in W_k$ with entries from $[0, t]$', is a continuous function $g : [0, t] \to W_k$ with $g(0) = o$ and $g(t) = y$. We denote this set by $\mathcal{R}_{k,t,y}$.

In [22] a mapping

$$\Phi : \mathcal{C}_{k,t} \to \bigcup_{y\in W_k} \mathcal{N}_{k,y} \times \mathcal{R}_{k,t,y}$$

is defined as continuous analogue of the Robinson–Schensted mapping. We refer the reader to that paper for a precise definition of Φ. Here we will simply recall some of its properties.

It is a one-to-one mapping, as in the discrete case. The natural measure on 'words' to consider in this setting is 'Brownian motion with drift'. Here we will restrict the discussion to the case of zero-drift, and consider standard Wiener measure $\mathbb{P}_t$ on $\mathcal{C}_{k,t}$. The Vandermonde function

$$h(x) = \prod_{i<j}(x_i - x_j)$$

is positive and harmonic on W_k and the corresponding Doob h-transform is well-defined; it can be interpreted as 'Brownian motion conditioned never to exit W_k'. It is possible (see, for example, [21]) to start this process at the origin: the measure

$$\mathbb{Q}_t[\mathrm{d}f] = \lim_{W_k\ni x\to o} \frac{h(f(t))}{h(x)} \mathbb{P}_t[\mathrm{d}f]$$

is well-defined. Let $\mathbb{U}_y$ denote the uniform probability measure on $\mathcal{N}_{k,y}$.

It is shown in [22] that

$$\mathbb{P}_t \circ \Phi^{-1}[\mathrm{d}n,\ \mathrm{d}f] = \mathbb{U}_{f(t)}[\mathrm{d}n]\mathbb{Q}_t[\mathrm{d}f]. \tag{90}$$

The connection with the Harish-Chandra/Itzykson–Zuber formula is as follows. Let $B = \{B(s), s \leqslant t\}$ be a standard Brownian motion in $\mathbb{R}^k$ and set $(N, R) = \Phi(B)$. Then the conditional distribution of $B(t)$, given $R(t)$, is the same as the conditional distribution of the *diagonal* of a $k \times k$ GUE random matrix, given its *eigenvalues*. The Laplace transform of this conditional distribution is given [12, 13] by the Harish-Chandra/Itzykson–Zuber formula: if we set

$$K(x, \mathrm{d}y) = \mathbb{P}[B(t) \in \mathrm{d}y | R(t) = x]$$

then

$$\int_{\mathbb{R}^k} e^{\theta \cdot y} K(x, \, dy) = \det(e^{\theta_i x_j})[h(x)h(\theta)]^{-1}.$$

Moreover, the transition kernels p_t and q_t associated with $\mathbb{P}_t$ and $\mathbb{Q}_t$ are intertwined via the Markov kernel K, that is

$$\int_{W_k} q_t(x, dy) K(y, \cdot) = \int_{\mathbb{R}^k} K(x, dy) p_t(y, \cdot)$$

for all $x \in W_k$.

For a discussion on the role of intertwining in the context of Pitman's $2M - X$ theorem (the case $k = 2$), see [26]. See also [8] for related work in a general symmetric spaces context.

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