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A PATH-TRANSFORMATION FOR RANDOM WALKS AND THE ROBINSON-SCHENSTED CORRESPONDENCE

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ABSTRACT. The author and Marc Yor recently introduced a path-transformation $G^{(k)}$ with the property that, for X belonging to a certain class of random walks on $\mathbb{Z}_+^k$, the transformed walk $G^{(k)}(X)$ has the same law as the original walk conditioned never to exit the Weyl chamber $\{x: x_1 \leq \cdots \leq x_k\}$. In this paper, we show that $G^{(k)}$ is closely related to the Robinson-Schensted algorithm, and use this connection to give a new proof of the above representation theorem. The new proof is valid for a larger class of random walks and yields additional information about the joint law of X and $G^{(k)}(X)$. The corresponding results for the Brownian model are recovered by Donsker's theorem. These are connected with Hermitian Brownian motion and the Gaussian Unitary Ensemble of random matrix theory. The connection we make between the path-transformation $G^{(k)}$ and the Robinson-Schensted algorithm also provides a new formula and interpretation for the latter. This can be used to study properties of the Robinson-Schensted algorithm and, moreover, extends easily to a continuous setting.

1. Introduction and summary

For $k \geq 2$, denote the set of probability distributions on $\{1, \dots, k\}$ by $\mathcal{P}_k$. Let $(\xi_m,\ m \geq 1)$ be a sequence of independent random variables with common distribution $p \in \mathcal{P}_k$ and, for $1 \leq i \leq k$, $n \geq 0$, set

$$X_i(n) = |\{1 \leq m \leq n : \ \xi_m = i\}|. \tag{1}$$

If $p_1 < \cdots < p_k$, there is a positive probability that the random walk $X = (X_1, \dots, X_k)$ never exits the Weyl chamber

$$W = \{x \in \mathbb{R}^k : \ x_1 \leq \cdots \leq x_k\}; \tag{2}$$

this is easily verified using, for example, the concentration inequality

$$P(|X(n) - np| > \varepsilon) \leq K e^{-c(\varepsilon)n},$$

where $c(\varepsilon)$ and K are finite positive constants.

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In [36], a certain path-transformation $G^{(k)}$ was introduced with the property that:

Theorem 1.1. *The law of the transformed walk $G^{(k)}(X)$ is the same, assuming $p_1 < \cdots < p_k$, as that of the original walk X conditioned never to exit W.*

We will recall the definition of $G^{(k)}$ in Section 2 below.

This was motivated by a desire to find a multi-dimensional generalisation of Pitman's representation for the three-dimensional Bessel process [37], and to understand some striking connections which were recently discovered by Baik, Deift and Johansson [4], Baryshnikov [5] and Gravner, Tracy and Widom [21], between oriented percolation and random matrices. For more background on this, see [33].

The proof of Theorem 1.1 given in [36] uses certain symmetry and reversibility properties of M/M/1 queues in series; consequently, the transformation $G^{(k)}$ has a "queueing-theoretic" interpretation.

In this paper we will show that the path-transformation $G^{(k)}$ is closely related to the Robinson-Schensted correspondence. More precisely, if $\lambda(n) = (\lambda_1(n) \geq \cdots \geq \lambda_k(n))$ denotes the shape of the Young tableaux obtained, when one applies the Robinson-Schensted algorithm with column-insertion, to the random word $\xi_1 \cdots \xi_n$, then (for any realisation of X)

$$(G^{(k)}(X))(n) = (\lambda_k(n), \ldots, \lambda_1(n)).$$

Immediately, this yields a new representation and formula for the Robinson-Schensted algorithm, and this formula has a queueing interpretation. We will use this representation to recover known, and perhaps not-so-well-known, properties of the Robinson-Schensted algorithm.

Given this connection, Theorem 1.1 can now be interpreted as a statement about the evolution of the shape $\lambda(n)$ of a certain randomly growing Young tableau. We give a direct proof of this result using properties of the Robinson-Schensted correspondence. This also yields more information about the joint law of X and $G^{(k)}(X)$, and dispenses with the condition $p_1 < \cdots < p_k$.

As in [36], the corresponding results for the Brownian motion model can be recovered by Donsker's theorem. The path-transformation $G^{(k)}$ extends naturally to a continuous setting and, given the connection with the Robinson-Schensted algorithm, the continuous version can now be regarded as a natural extension of the Robinson-Schensted algorithm to a continuous setting. As discussed in [36], the results for Brownian motion have an interpretation in random matrix theory. In particular, Theorem 1.1 yields a representation for the eigenvalue process associated with Hermitian Brownian motion as a certain path-transformation (the continuous analogue of $G^{(k)}$) applied to a standard Brownian motion. The new results presented in this paper also yield new results in this context. This random matrix connection comes from the well-known fact that the eigenvalue process associated with Hermitian Brownian motion can be interpreted as Brownian motion conditioned never to exit the Weyl chamber W. We remark that a similar representation for the eigenvalues of Hermitian Brownian motion was independently obtained by Bougerol and Jeulin [11], in a more general context, by completely different methods.

The outline of the paper is as follows. In the next section we recall the definition of $G^{(k)}$ and record some of its properties. In section 3, we make the connection with the Robinson-Schensted algorithm, and briefly consider some immediate implications of this connection. A worked example is presented in section 4. In section 5, we record some properties of the conditioned walk of Theorem 1.1 and extend its definition beyond the case $p_1 < \cdots < p_k$. In section 6, we prove a generalisation of Theorem 1.1, in the context of Young tableaux, using properties of the RS correspondence. In section 7, we define a continuous version of the path-transformation and present the "Poissonized" analogues of the results of the previous section. In section 8, we present the corresponding results for the Brownian model, and briefly discuss the connection with random matrices. An application in queueing theory is presented in section 9, and we conclude the paper with some remarks in section 10.

Some notation: Let $b = \{e_1, \ldots, e_k\}$ denote the standard basis elements in $\mathbb{R}^k$. For $x, y \in \mathbb{R}_+^k$ we will write $x^y = x_1^{y_1} \cdots x_k^{y_k}$, $xy = (x_1y_1, \ldots, x_ky_k)$, $|x| = \sum_i x_i$ and define $x^* \in \mathbb{R}_+^k$ by $x_i^* = x_{k-i+1}$. Denote the origin in $\mathbb{R}^k$ by o.

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2. The path-transformation

The support of the random walk X, which we denote by Π_k, consists of paths $x : \mathbb{Z}_+ \to \mathbb{Z}_+^k$ with $x(0) = 0$ and, for each $n > 0$, $x(n) - x(n-1) \in b$. Let Π_k^W denote the subset of those paths taking values in W. It is convenient to introduce another set Λ_k of paths $x : \mathbb{Z}_+ \to \mathbb{Z}_+^k$ with $x(0) = 0$ and $x(n) - x(n-1) \in \{0, e_1, \ldots, e_k\}$, for each $n > 0$.

For $x, y \in \Lambda_1$, define $x \triangle y \in \Lambda_1$ and $x \triangledown y \in \Lambda_1$ by

$$(x \triangle y)(n) = \min_{0 \le m \le n} [x(m) + y(n) - y(m)] \tag{3}$$

and

$$(x \triangledown y)(n) = \max_{0 \le m \le n} [x(m) + y(n) - y(m)]. \tag{4}$$

The operations $\triangle$ and $\triangledown$ are not associative in general. Unless otherwise delineated by parentheses, the default order of operations is from left to right; for example, when we write $x \triangle y \triangle z$, we mean $(x \triangle y) \triangle z$.

The mappings $G^{(k)} : \Lambda_k \to \Lambda_k$ are defined as follows. Set

$$G^{(2)}(x, y) = (x \triangle y, y \triangledown x) \tag{5}$$

and, for $k > 2$,

$$\begin{aligned} G^{(k)}(x_1, \ldots, x_k) = (&x_1 \triangle x_2 \triangle \cdots \triangle x_k, \\ &G^{(k-1)}(x_2 \triangledown x_1, x_3 \triangledown (x_1 \triangle x_2), \ldots, x_k \triangledown (x_1 \triangle \cdots \triangle x_{k-1}))). \end{aligned} \tag{6}$$

Note that $G^{(k)} : \Pi_k \to \Pi_k^W$.

We will now give an alternative definition of $G^{(k)}$ which will be useful for making the connection with the Robinson-Schensted correspondence.

Occasionally, we will suppress the dependence of functions on x, when the context is clear: for example, we may write $G^{(k)}$ instead of $G^{(k)}(x)$, and so on.

For $k \geq 2$, define maps $D^{(k)} : \Lambda_k \to \Lambda_k$ and $T^{(k)} : \Lambda_k \to \Lambda_{k-1}$ by

$$D^{(k)}(x) = (x_1, x_1 \triangle x_2, \ldots, x_1 \triangle \cdots \triangle x_k) \tag{7}$$

and

$$T^{(k)}(x) = (x_2 \triangledown x_1, x_3 \triangledown (x_1 \triangle x_2), \ldots, x_k \triangledown (x_1 \triangle \cdots \triangle x_{k-1})). \tag{8}$$

For notational convenience, let $D^{(1)}$ be the identity transformation.

Note that the above definition is recursive: for $i \geq 2$,

$$D_i^{(k)} = D_{i-1}^{(k)} \triangle x_i \tag{9}$$

and

$$T_{i-1}^{(k)} = x_i \triangledown D_{i-1}^{(k)}. \tag{10}$$

Alternatively, we can write

$$(D_i^{(k)}, T_{i-1}^{(k)}) = G^{(2)}(D_{i-1}^{(k)}, x_i). \tag{11}$$

For each $x \in \Lambda_k$, consider the triangular array of sequences $d^{(i)} \in \Lambda_{k-i+1}$, $1 \leq i \leq k$, defined as follows. Set

$$d^{(1)} = D^{(k)}(x), \quad t^{(1)} = T^{(k)}(x),$$

$$d^{(2)} = D^{(k-1)}(t^{(1)}), \quad t^{(2)} = T^{(k-1)}(t^{(1)}),$$

and so on; for $i \leq k$,

$$d^{(i)} = D^{(k-i+1)}(t^{(i-1)}),$$

and for $i \leq k-1$,

$$t^{(i)} = T^{(k-i+1)}(t^{(i-1)}).$$

Recalling the definition of $G^{(k)}$ given earlier, we see that

$$G^{(k)} = (d_k^{(1)}, \ldots, d_1^{(k)}). \tag{12}$$

Note also that, for each $i \leq k$,

$$G^{(i)}(x_1, \ldots, x_i) = (d_i^{(1)}, \ldots, d_1^{(i)}). \tag{13}$$

We will conclude this section by recording some useful properties and interpretations of the operations $\triangle$ and $\triangledown$, and of the path-transformation $G^{(k)}$, for later reference. We defer the proofs: these will be given in the appendix.

The following notation for increments of paths will be useful: for $x \in \Lambda_k$ and $l \geq n$, set $x(n,l) = x(l) - x(n)$.

The operations $\triangle$ and $\triangledown$ have a queueing-theoretic interpretation, which we will make strong use of when we make the connection with the Robinson-Schensted correspondence in the next section. For more general discussions on "min-plus algebra" and queueing networks, see [1].

Suppose $(x, y) \in \Pi_2$, and consider a simple queue which evolves as follows. At each time n, either $x(n) - x(n-1) = 1$ and $y(n) - y(n-1) = 0$, in which case a new customer arrives at the queue, or $x(n) - x(n-1) = 0$ and $y(n) - y(n-1) = 1$, in which case, if the queue is not empty, a customer departs (otherwise nothing happens). The number of customers remaining in the queue at time n, which we denote by $q(n)$, satisfies the Lindley recursion

$$q(n) = \max\{q(n-1) + \epsilon(n), 0\}, \tag{14}$$

where $\epsilon(n) = x(n) - x(n-1) - y(n) + y(n-1)$. Iterating (14), we obtain

$$q(n) = \max_{0 \le m \le n} [x(m,n) - y(m,n)]. \tag{15}$$

Thus, the number of customers $d(n)$ to depart up to and including time n is given by

$$d(n) = x(n) - q(n) = (x \triangle y)(n). \tag{16}$$

We also have

$$t(n) := x(n) + u(n) = (y \triangledown x)(n), \tag{17}$$

where

$$u(n) = y(n) - d(n)$$

is the number of times $m \le n$ that $y(m) - y(m-1) = 1$ and $q(m-1) = 0$; in the language of queueing theory, $u(n)$ is the number of "unused services" up to and including time n. (For this queue we refer to the points of increase of y as "services".)

Lemma 2.1. *For* $(x, y) \in \Lambda_2$,

$$x \triangle y + y \triangledown x = x + y \tag{18}$$

and, if $\min_{l \ge n}[x(l) - (x \triangle y)(l)] = 0$,

$$\begin{aligned} x(n) - (x \triangle y)(n) &= \max_{0 \le m \le n} [x(m,n) - y(m,n)] \\ &= \max_{l \ge n} [(x \triangle y)(n,l) - (y \triangledown x)(n,l)]. \end{aligned}$$

In particular, writing $G^{(2)} \equiv G^{(2)}(x,y)$, *we have*

$$(x(n), y(n)) = G^{(2)}(n) + F^{(2)}\left(G^{(2)}(n,l),\ l \ge n\right), \tag{19}$$

where $F^{(2)} : \mathcal{D} \to \mathbb{Z}^2$ *is defined on*

$$\mathcal{D} = \{z \in (\mathbb{Z}^2)^{\mathbb{Z}_+} :\ M(z) = \max_{n \ge 0}[z_1(n) - z_2(n)] < \infty\}$$

by $F^{(2)}(z) = (M(z), -M(z))$.

In the queueing context described above, Lemma 2.1 states that $x + y = d + t$ and, if $\min_{l \ge n} q(l) = 0$,

$$q(n) = \max_{m \ge n}[d(n,m) - t(n,m)]. \tag{20}$$

The first identity is readily verified. The formula for $q(n)$ in terms of the future increments of d and t follows from the time-reversal symmetry in the dynamics of the system: this formula is the dual of (15). When time is reversed, the roles played by (x, y) and (d, t) are interchanged. This symmetry is at the heart of the proof of Theorem 1.1 given in [36], where it is considered in an equilibrium context.

Note that, if we set $z = y - x$ and $s(n) = \max_{0 \le m \le n} z(m)$, then

$$y \triangledown x - x \triangle y = 2s - z$$

and (20) is equivalent to the well-known identity

$$s(n) = \min_{l \ge n}[2s(l) - z(l)].$$

This is familiar in the context of Pitman's representation for the three-dimensional Bessel process. Observe that the statement of Theorem 1.1 in the case $k = 2$ is equivalent to the following discrete version of Pitman's theorem: if $\{Z(n), n \geq 0\}$ is a simple random walk on $\mathbb{Z}$ with positive drift, started at 0, and we set $S(n) = \max_{0 \leq m \leq n} Z(n)$, then $2S - Z$ has the same law as that of Z conditioned to stay nonnegative. The usual statement of Pitman's theorem can be recovered from Theorem 8.1 below.

Lemma 2.1 has the following generalisation:

Lemma 2.2. *For $x \in \Lambda_k$, writing $G^{(k)} \equiv G^{(k)}(x)$, we have*

$$|G^{(k)}| = |x| \tag{21}$$

and, if $\min_{l \geq n}(d_j^{(i)} - d_{j+1}^{(i)})(l) = 0$, for $1 \leq j < i < k \quad (c_n)$,

$$x(n) = G^{(k)}(n) + F^{(k)}\left(G^{(k)}(n,l),\ l \geq n\right), \tag{22}$$

where the function $F^{(k)}$ will be defined in the proof.

As we remarked earlier, the operations $\triangle$ and $\triangledown$ are not associative. The following identities are useful for manipulating complex combinations of these operations.

Lemma 2.3. *For $(a, b, c) \in \Lambda_3$ we have*

$$a \triangledown (c \triangle b) \triangledown (b \triangledown c) = a \triangledown b \triangledown c \tag{23}$$

and

$$a \triangle (c \triangledown b) \triangle (b \triangle c) = a \triangle b \triangle c. \tag{24}$$

For example, (23) immediately yields

Lemma 2.4. *For $x \in \Lambda_k$, $G_k^{(k)}(x) = x_k \triangledown \cdots \triangledown x_1$.*

3. Connection with the Robinson-Schensted algorithm

We refer the reader to the books of Fulton [18] and Stanley [40] for detailed discussions on the Robinson-Schensted algorithm and its properties. The standard Robinson-Schensted algorithm takes a word $w = a_1 \cdots a_n \in \{1, 2, \ldots, k\}^n$ and proceeds, by "row-inserting" the numbers a_1, then a_2, and so on, to construct a semistandard tableau $P(w)$ associated with w, of size n with entries from the set $\{1, 2, \ldots, k\}$. If one also maintains a "recording tableau" $Q(w)$, which is a standard tableau of size n, the mapping from words $\{1, 2, \ldots, k\}^n$ to pairs of semistandard and standard tableaux of size n, the semistandard tableau having entries from $\{1, 2, \ldots, k\}$ and both having the same shape, is a bijection: this is the Robinson-Schensted correspondence. One can also do all of the above using "column-insertion" instead of row-insertion to construct the semistandard tableau, but still maintaining a recording tableau, and the resulting map is also a bijection. Column and row insertion are not the same thing, but they are related in the following way: the semistandard tableau obtained by applying the Robinson-Schensted algorithm, with column-insertion, to the word $a_1 \cdots a_n$ is the same as the one obtained by applying the Robinson-Schensted algorithm, with row-insertion, to the reversed word $a_n \cdots a_1$. The standard tableaux obtained in each case are also related, but we do not need this and refer the reader to [18] for details.

Fix $x \in \Pi_k$, let $d^{(i)} \in \Lambda_{k-i+1}$, $1 \leq i \leq k$, be the corresponding triangular array of sequences defined in the previous section.

For each n, construct a semistandard Young tableau as follows. In the first row, put

$$d_1^{(1)}(n)\ 1\text{'s},\ d_1^{(2)}(n) - d_1^{(1)}(n)\ 2\text{'s},\ \ldots d_1^{(k)}(n) - d_1^{(k-1)}(n)\ k\text{'s};$$

in the second row, put

$$d_2^{(1)}(n)\ 2\text{'s},\ d_2^{(2)}(n) - d_2^{(1)}(n)\ 3\text{'s},\ \ldots, d_2^{(k-1)}(n) - d_2^{(k-2)}(n)\ k\text{'s},$$

and so on. In the final row, there are just $d_k^{(1)}(n)$ k's. Denote this tableau by $\tau(n)$. For example, if $k = 3$ and

$$\begin{matrix} d_1^{(1)}(7) & d_2^{(1)}(7) & d_3^{(1)}(7) \\ & d_1^{(2)}(7) & d_2^{(2)}(7) \\ & & d_1^{(3)}(7) \end{matrix} \quad = \quad \begin{matrix} 2 & 2 & 1 \\ & 3 & 2 \\ & & 4 \end{matrix} \tag{25}$$

then the corresponding semistandard tableau $\tau(7)$ is

(26)

1	1	2	3
2	2		
3			

Let a_m be the sequence defined by $a_m = i$ whenever

$$x(m) - x(m-1) = e_i.$$

Theorem 3.1. *The semistandard tableau $\tau(n)$ is precisely the one that is obtained when one applies the Robinson-Schensted algorithm, with column insertion, to the word $a_1 \cdots a_n$. In particular, if $l(n)$ denotes the shape of $\tau(n)$, then $l(n)^* = (G^{(k)}(x))(n)$.*

Proof. It will suffice to describe how the mapping $G^{(k)}$ acts on a typical element of Π_k, from an algorithmic point of view.

For each $k \geq 2$, the maps $D^{(k)} : \Lambda_k \to \Lambda_k$ and $T^{(k)} : \Lambda_k \to \Lambda_{k-1}$ can be defined as follows. Fix $x \in \Pi_k$ and set $d = D^{(k)}(x)$, $t = T^{(k)}(x)$. Set $d(0) = t(0) = 0$, and define the sequences $d(n)$ and $t(n)$ inductively on n. Suppose $x(n) - x(n-1) = e_i$; that is, $a_n = i$. We need to treat the cases $i = 1$ and $i = k$ separately.

Suppose $i = 1$. Then we set $d(n) = d(n-1) + e_1$ and $t(n) = t(n-1) + e_1$.

If $i = k$, and $d_k(n-1) < d_{k-1}(n-1)$, we set $d(n) = d(n-1) + e_k$ and $t(n) = t(n-1)$.

If $i = k$, and $d_k(n-1) = d_{k-1}(n-1)$, we set $d(n) = d(n-1)$ and $t(n) = t(n-1) + e_{k-1}$.

Now suppose $1 < i < k$. If $d_i(n-1) < d_{i-1}(n-1)$, set $d(n) = d(n-1) + e_i$ and $t(n) = t(n-1) + e_i$; if $d_i(n-1) = d_{i-1}(n-1)$, set $d(n) = d(n-1)$ and $t(n) = t(n-1) + e_{i-1}$. Recall that $D^{(1)}$ is the identity transformation.

In queueing language, we have just constructed a series of k queues in tandem. Initially there are infinitely many customers in the first queue and the other queues are all empty. At each time n, if $x_i(n+1) - x_i(n) = 1$ (or, equivalently, $a_n = i$) there is a "service event" at the i^{th} queue; if this queue is not empty a customer departs from it and, if $i < k$, joins the $(i+1)^{th}$ queue. The number of departures from the i^{th} queue up to and including time n is given by $d_i(n)$ and $t_i(n) = d_i(n) + u_i(n)$, where $u_i(n)$ is the number of "unused services" at the $(i+1)^{th}$ queue up to and including time n. Recalling the queueing-theoretic interpretations of $\triangle$ and $\triangledown$, we see that $d_i = x_1 \triangle \cdots \triangle x_i$ and $t_i = x_i \triangledown d_{i-1}$.

Now fix $x \in \Pi_k$, and recall the definition of the triangular array of sequences $d^{(i)} \in \Lambda_{k-i+1}$, $1 \leq i \leq k$. Set

$$d^{(1)} = D^{(k)}(x), \quad t^{(1)} = T^{(k)}(x),$$

$$d^{(2)} = D^{(k-1)}(t^{(1)}), \quad t^{(2)} = T^{(k-1)}(t^{(1)}),$$

and so on; for $i \leq k$,

$$d^{(i)} = D^{(k-i+1)}(t^{(i-1)}),$$

and for $i \leq k-1$,

$$t^{(i)} = T^{(k-i+1)}(t^{(i-1)}).$$

Here we have constructed a "series of queues in series", the entire system "driven" by x. This is represented in Figure 1 for the case $k = 3$.

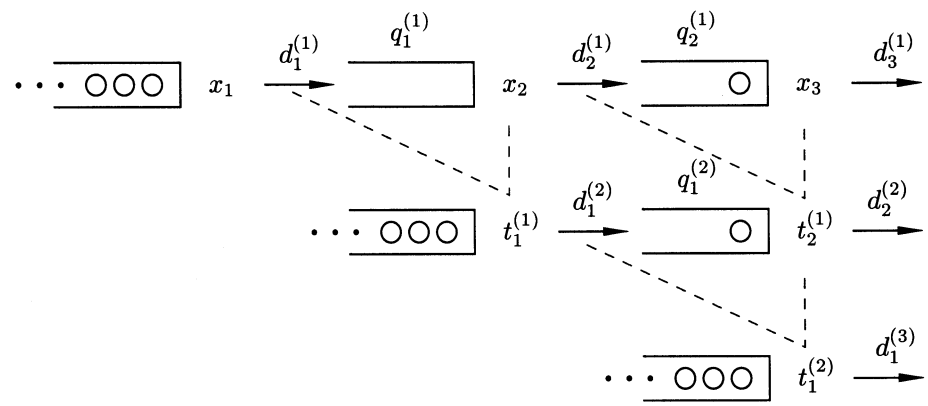

FIGURE 1. The series of queues in series ($k = 3$)

The first series of queues is just the one described above: there are k queues in the series, and initially queues 2 thru k are empty and the first queue has infinitely many customers; whenever x_i increases by one there is a service at the i^{th} queue and one customer is permitted to depart (and proceed to the next queue if $i < k$). The number of departures from the i^{th} queue up to time n is given by $d_i^{(1)}(n)$.

The second series of queues has $t^{(1)}$ "moving" the customers in place of x. This time there are $k-1$ queues. Initially, queues 2 thru $k-1$ are empty and the first queue has infinitely many customers. There is a service event at the i^{th} queue whenever $t_i^{(1)}$ increases by one—that is, whenever, in the first series, there is a departure from the i^{th} queue or an unused service at the $(i+1)^{th}$ queue (these events will never occur simultaneously).

The second series generates a new sequence of t's, which we denote by $t^{(2)}$, and this is used to drive the third series, which consists of $k-2$ queues, and so on.

It is useful to define

$$q_i^{(j)} = d_i^{(j)} - d_{i+1}^{(j)}; \tag{27}$$

$q_i^{(j)}(n)$ is just the number of customers in the $(i+1)^{th}$ queue of the j^{th} series at time n. For example, in the network shown in Figure 1, $q_1^{(1)} = 0$ and $q_2^{(1)} = q_1^{(2)} = 1$.

Now consider the evolution of the corresponding semistandard tableaux $\tau(n)$, $n \geq 1$. See Figure 2. Another look at the algorithm described above should

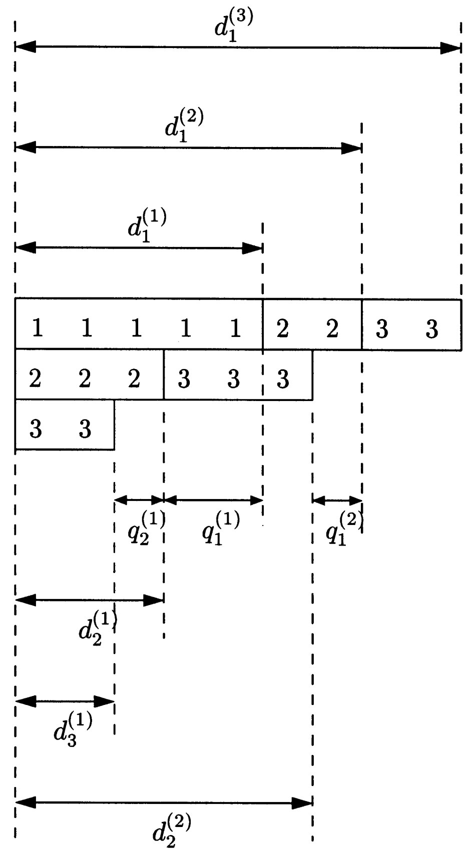

FIGURE 2. The tableau $\tau(17)$

convince the reader that $\tau(n)$ is precisely the semistandard tableau obtained when one applies the Robinson-Schensted algorithm, with *column* insertion, to the word $a_1 \cdots a_n$.

To see this, look at the tableau $\tau(17)$ represented in Figure 2. Recall that $a_m = i$ if x_i increases by one at time m, in which case there is a service at the i^{th} queue of the first series.

Suppose that the next "letter" $a(18) = 2$. Since $d_2^{(1)} < d_1^{(1)}$, that is, $q_1^{(1)} > 0$, we have a departure from the second queue in the first series; that is, we decrease $q_1^{(1)}$ by one and increase $d_2^{(1)}$ by one. In turn, this leads to an increase in $t_2^{(1)}$, that is, a service at the second queue in the second series, and so, since $q_1^{(2)} > 0$, we decrease $q_1^{(2)}$ by one, and increase $d_2^{(2)}$ by one. That's it; the resulting tableau $\tau(18)$ is

1	1	1	1	1	2	2	3	3
2	2	2	2	3	3	3		
3	3							

and we recognise this procedure as column-insertion of the number 2 into the tableau $\tau(17)$.

Note that, now, $q_1^{(2)} = 0$. Suppose that the next letter $a(19)$ is also a 2. We still have $q_1^{(1)} > 0$; so we decrease $q_1^{(1)}$ by one and increase $d_2^{(1)}$ by one. In turn, this leads to an increase in $t_2^{(1)}$, that is, a service at the second queue in the second series; but $q_1^{(2)} = 0$, and so this service is unused and there is no departure (that is, no increase in $d_2^{(2)}$). The unused service leads to an increase in $t_1^{(3)}$, that is, a service at the only queue in the third series, and we increase $d_1^{(3)}$ by one. The resulting tableau $\tau(19)$ is

1	1	1	1	1	2	2	3	3	3
2	2	2	2	2	3	3			
3	3								

and, again, we recognise this procedure as column-insertion of the number 2 into the tableau $\tau(18)$, and so on. □

We will give a completely worked example, starting from an empty tableau, in the next section.

We conclude this section with some remarks on the immediate implications of Theorem 3.1. Let $l(n)$ and $\alpha(n)$ respectively denote the shape and weight of $\tau(n)$. In view of Theorem 3.1, Lemma 2.4 can now be interpreted as stating that $l_1(n)$ is the length of the longest non-decreasing subsequence in the reversed word $a_n \cdots a_1$. This is a well-known property of the Robinson-Schensted algorithm. Thus, Lemma 2.4 can be regarded as a corollary of Theorem 3.1, or the proof of Lemma 2.4 given in the appendix can be regarded as a new proof of the longest increasing subsequence property of the Robinson-Schensted algorithm.

More generally, we can compare the statement of Theorem 3.1 with Greene's theorem, and this leads to some remarkable identities. Greene's theorem (see, for example, [29]) states that, if $m_i(n)$ denotes the maximum of the sum of the lengths of i disjoint, non-decreasing subsequences in the reversed word $a_n \cdots a_1$, then, for $i \le k$,

$$m_i(n) = l_1(n) + l_2(n) + \cdots + l_i(n). \tag{28}$$

It therefore follows from Theorem 3.1 that

$$m_i(n) = G_k^{(k)}(n) + G_{k-1}^{(k)}(n) + \cdots + G_{k-i+1}^{(k)}(n). \tag{29}$$

It would be interesting to see a direct proof of this identity. Similarly, one can compare Theorem 3.1 with the various extensions of Greene's theorem given, for example, in [6].

The implications of Lemma 2.2 for the Robinson-Schensted algorithm would appear to be less well known. Fix $k \ge 2$, and set $H(z) = F^{(k)}(z^*)^*$.

Corollary 3.2. *Suppose that (c_n) holds. The weight $\alpha(n)$ of $\tau(n)$ can be recovered from the sequence of shapes*

$$\{\operatorname{sh} \tau(l),\ l \ge n\}.$$

In fact, if we set, for $m \ge 0$,

$$u(m) = \operatorname{sh} \tau(n+m) - \operatorname{sh} \tau(n),$$

then

$$\alpha(n) = \operatorname{sh} \tau(n) + H(u).$$

Recall that the recording tableaux $\sigma(n)$ are nested; the limiting standard tableau $\sigma(\infty) = \lim_{n\to\infty} \sigma(n)$ is thus a well-defined object. It follows from Corollary 3.2 that provided (c_n) holds we can recover the infinite word $a_1 a_2 \ldots$ from $\sigma(\infty)$.

This is also true for the Robinson-Schensted algorithm applied with row insertion. To see this, recall that the recording tableaux maintained when one applies the row insertion algorithm to the infinite word $a_1 a_2 \ldots$ is the same as that maintained when one applies the column insertion algorithm to the word $a_1^\dagger a_2^\dagger \ldots$, where $a_n^\dagger = k - a_n + 1$ (or see, for example, [18, A.2, Exercise 5]).

4. A worked example

Suppose $k = 3$ and $n = 7$, and we apply the Robinson-Schensted algorithm with column insertion to the word $a_1 \cdots a_7 = 3112322$. We obtain the following sequence of semistandard tableaux:

$$\begin{array}{|c|}\hline 3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&3\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}$$

The evolution of the corresponding queueing network is as follows. For each n, set

$$Q(n) = \begin{bmatrix} q_1^{(1)}(n) & q_2^{(1)}(n) \\ & q_1^{(2)}(n) \end{bmatrix}$$

and

$$D(n) = \begin{bmatrix} d_1^{(1)}(n) & d_2^{(1)}(n) & d_3^{(1)}(n) \\ & d_1^{(2)}(n) & d_2^{(2)}(n) \\ & & d_1^{(3)}(n) \end{bmatrix}.$$

Initially,

$$Q(0) = \begin{bmatrix} 0 & 0 \\ & 0 \end{bmatrix} \quad \text{and} \quad D(0) = \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}.$$

At time 1, $a_1 = 3$; so there is a service at the third queue in the first series. Since $q_2^{(1)}(0) = 0$, this queue is empty, and the service is unused, leading to an increase in $t_2^{(1)}$ and hence a service at the second queue in the second series. This is also unused; so we have a service at the first (and only) queue in the third series, and there is a departure there: $d_1^{(3)}(1) = 1$. Thus,

$$Q(1) = \begin{bmatrix} 0 & 0 \\ & 0 \end{bmatrix} \quad \text{and} \quad D(1) = \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{bmatrix}.$$

The corresponding tableau $\tau(1)$ is

$$\begin{array}{|c|}\hline 3\\\hline\end{array}$$

At time 2, $a_2 = 1$; so there is a service at the first queue in the first series and a customer departs to join the second queue: thus, $d_1^{(1)}(2) = 1$ and $q_1^{(1)}(2) = 1$. In the second series, this leads to an increase in $t_1^{(1)}$ and hence a departure from the first queue to the second queue: thus, $d_1^{(2)}(2) = 1$ and $q_1^{(2)}(2) = 1$. In turn, this leads to an increase in $t_1^{(2)}$ and hence a service at the only queue in the third series, which yields a (second) departure from that queue and we have $d_1^{(3)}(2) = 2$. Thus,

$$Q(2) = \begin{bmatrix} 1 & 0 \\ & 1 \end{bmatrix} \quad \text{and} \quad D(2) = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 2 \end{bmatrix}.$$

The corresponding tableau $\tau(2)$ is

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline \end{array}$$

Similarly, at time 3, $a_3 = 1$, and we get

$$Q(3) = \begin{bmatrix} 2 & 0 \\ & 2 \end{bmatrix} \quad \text{and} \quad D(3) = \begin{bmatrix} 2 & 0 & 0 \\ & 2 & 0 \\ & & 3 \end{bmatrix}.$$

The corresponding tableau $\tau(3)$ is

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline \end{array}$$

At time 4, $a_4 = 2$, and there is a departure from the second queue in the first series. This provides a service at the second queue in the second series, which is nonempty; so we have a departure from that queue as well. Thus,

$$Q(4) = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \quad \text{and} \quad D(4) = \begin{bmatrix} 2 & 1 & 0 \\ & 2 & 1 \\ & & 3 \end{bmatrix}.$$

The corresponding tableau $\tau(4)$ is

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

At time 5, there is a service at the third queue in the first series, and a customer departs. That's all. Thus,

$$Q(5) = \begin{bmatrix} 1 & 0 \\ & 1 \end{bmatrix} \quad \text{and} \quad D(5) = \begin{bmatrix} 2 & 1 & 1 \\ & 2 & 1 \\ & & 3 \end{bmatrix}.$$

The corresponding tableau $\tau(5)$ is

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}$$

At time 6, there is a service at the second queue in the first series, and a customer departs; this yields a service at the second queue in the second series and a customer departs from there also. Thus,

$$Q(6) = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix} \quad \text{and} \quad D(6) = \begin{bmatrix} 2 & 2 & 1 \\ & 2 & 2 \\ & & 3 \end{bmatrix}.$$

The corresponding tableau $\tau(6)$ is

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

At time 7, there is a service at the second queue in the first series, but this queue is empty and it is not used; this yields a service at the *first* queue in the second series, leading to a departure from that queue and consequently a service at (and departure from) the first (and only) queue in the third series. Thus,

$$Q(7) = \begin{bmatrix} 0 & 1 \\ & 1 \end{bmatrix} \quad \text{and} \quad D(7) = \begin{bmatrix} 2 & 2 & 1 \\ & 3 & 2 \\ & & 4 \end{bmatrix}.$$

The final tableau $\tau(7)$ is

1	1	2	3
2	2		
3			

5. Random walk in a Weyl chamber

In this section we record some properties of the conditioned walk of Theorem 1.1, and extend its definition beyond the case $p_1 < \cdots < p_k$.

The random walk X is a Markov chain on $\mathbb{Z}_+^k$ with $X(0) = o$ and transition matrix

$$P(x, y) = p^{y-x} 1_{\{y-x \in b\}}.$$

Denote by P_x the law of the walk started at $x \in W \cap \mathbb{Z}_+^k$.

We will refer to the random walk with $p_1 = \cdots = p_k$ as the *homogeneous walk.*

Denote by s_l the Schur polynomial associated with the integer partition $l_1 \geq l_2 \geq \cdots l_k \geq 0$.

Lemma 5.1. *For any $r \in \mathcal{P}_k$, the function $h_r : \mathbb{Z}_+^k \to \mathbb{R}$, defined by*

$$h_r(x) = p^{-x} s_{x^*}(r) 1_{x \in W},$$

is harmonic for P. Note that h_r is strictly positive on $W \cap \mathbb{Z}_+^k$.

Proof. This follows immediately from the identity

$$\sum_i s_{l+e_i}(r) = s_l(r),$$

which in turn can be seen as a special case of the Weyl character formula, or can be verified directly using the formula

$$s_l(p) = \det\left(p_i^{l_j+k-j}\right) / \det\left(p_i^{k-j}\right).$$

□

Lemma 5.2. *Suppose $0 < p_1 < \cdots < p_k$. Then, for any $x \in W \cap \mathbb{Z}_+^k$,*

$$P_x(X(n) \in W, \text{ for all } n \geq 0) = C p^{-x} s_{x^*}(p),$$

where C is a constant independent of x. In particular, the transition matrix associated with the conditioned walk of Theorem 1.1 is given by

$$\hat{P}(x, y) = \frac{p^{-y} s_{y^*}(p)}{p^{-x} s_{x^*}(p)} P(x, y) = \frac{s_{y^*}(p)}{s_{x^*}(p)} 1_{\{y-x \in b\}}. \tag{30}$$

Proof. When $r = p$, the Doob transform of the random walk X via the harmonic function h_r has transition matrix $\hat{P}$. Note that this can also be regarded as the Doob transform of the homogeneous walk via the function $x \mapsto s_{x^*}(kp)$. It follows from the asymptotic analysis of the Green function associated with the (Poissonized) homogeneous walk presented in [28] that, if $\kappa(x, y)$ is the Martin kernel associated with the homogeneous walk, then

$$\kappa(x, y) \to \text{constant} \times s_{x^*}(kp)$$

whenever y tends to infinity in W in the direction p. Thus, by standard Doob-Hunt theory (see, for example, [15], [43]), any realisation of the corresponding Doob transform, starting from the origin o, almost surely goes to infinity in the direction p. Moreover, any Doob transform on W which almost surely goes to infinity in the direction p is necessarily the same Doob transform. It therefore suffices to show that the "properly" conditioned walk of Theorem 1.1, which is the Doob transform of X via the harmonic function

$$g(x) = P_x(X(n) \in W, \text{ for all } n \geq 0),$$

almost surely goes to infinity in the direction p. But this follows immediately from the estimate, denoting the law of the conditioned walk by $\hat{P}$,

$$\hat{P}(|X(n) - pn| > \epsilon n) \leq P(|X(n) - pn| > \epsilon n)/g(x) \leq K e^{-c(\epsilon) n}/g(x),$$

where $c(\epsilon) > 0$, and a standard Borel-Cantelli argument. □

Note that the transition matrix $\hat{P}$ is well-defined by (30) for any $p \in \mathcal{P}_k$ and, by the symmetry of the Schur polynomials, is symmetric in the p_i.

The proof of Lemma 5.2 given above is presented in more detail in [34], where an explicit formula for the constant C is also given.

6. The Robinson-Schensted algorithm with random words

Having made the connection between the path-transformation $G^{(k)}$ and the Robinson-Schensted algorithm, we will now give a direct proof of Theorem 1.1, purely in the latter context. In fact, we will present a more general result, which does not require the condition $p_1 < \cdots < p_k$.

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables with common distribution $p \in \mathcal{P}_k$. Let $(S(n), T(n))$ be the pair of semistandard and standard tableaux associated, by the Robinson-Schensted correspondence (with column-insertion), with the random word $\xi_1 \xi_2 \cdots \xi_n$. Here, $T(n)$ is the recording tableau. Denote the shape of $S(n)$ by $\lambda(n)$; the weight (or type) of $S(n)$ is $X(n)$, where

$$X_i(n) = |\{1 \leq m \leq n : \ \xi_m = i\}|.$$

Note that X is the random walk discussed throughout this paper, with transition matrix

$$P(x, y) = p^{y-x} 1_{\{y-x \in b\}}.$$

The joint law of $(S(n), T(n))$ is given, for $\operatorname{sh} \sigma = \operatorname{sh} \tau \vdash n$, by

$$P(S(n) = \sigma, T(n) = \tau) = p^{\sigma}, \tag{31}$$

and, for $x \in C = \{x \in \mathbb{Z}_+^k : \ x_1 \geq \cdots \geq x_k\}$,

$$P(\lambda(n) = x) = s_x(p) f_x.$$

Here p^σ is shorthand for p^a, where a is the weight of the tableau σ, and f_x is the number of standard tableaux with shape x. The formula (31) follows immediately from the fact that the Robinson-Schensted correspondence with column-insertion, as in the case with row-insertion, is bijective.

Consider the Doob transform of P on C, defined by its transition matrix

$$Q(x,y) = \frac{p^{-y}s_y(p)}{p^{-x}s_x(p)}P(x,y) = \frac{s_y(p)}{s_x(p)}1_{\{y-x\in b\}}. \tag{32}$$

Theorem 6.1. *λ is a Markov chain on C with transition matrix Q.*

Proof. For $x, y \in C$, we will write $x \nearrow y$ if $y-x \in b$. Recall that a standard tableau τ with entries $\{1, 2, \dots, n\}$ can be identified with a sequence of integer partitions

$$l(1) \nearrow l(2) \nearrow \cdots \nearrow l(n),$$

where $l(m)$ is the shape of the subtableau of τ consisting only of the entries $\{1, 2, \dots, m\}$. Since $T(n)$ is a recording tableau, it is identified in this way with the sequence

$$\lambda(1) \nearrow \lambda(2) \nearrow \cdots \nearrow \lambda(n).$$

Thus, summing (31) over semistandard tableaux σ with a given shape $l(n) \vdash n$, we obtain

$$P(\lambda(1) = l(1), \dots, \lambda(n) = l(n)) = \sum_{\mathrm{sh}\,\sigma = l(n)} p^\sigma = s_{l(n)}(p), \tag{33}$$

and so, for $x \nearrow y \vdash n+1$,

$$\begin{aligned} &P(\lambda(n+1) = y \mid \lambda(1) = l(1), \dots, \lambda(n-1) = l(n-1), \lambda(n) = x) \\ &= \frac{P(\lambda(1) = l(1), \dots, \lambda(n-1) = l(n-1), \lambda(n) = x, \lambda(n+1) = y)}{P(\lambda(1) = l(1), \dots, \lambda(n-1) = l(n-1), \lambda(n) = x)} \\ &= \frac{s_y(p)}{s_x(p)}, \end{aligned}$$

as required. □

Recalling the connection between $G^{(k)}$ and the Robinson-Schensted algorithm described in the previous section, and comparing Q with the transition matrix $\hat{P}$ defined by (30), we deduce the following generalisation of Theorem 1.1.

Corollary 6.2. *$\hat{X} = G^{(k)}(X)$ is a Markov chain on $W \cap \mathbb{Z}_+^k$ with transition matrix $\hat{P}$.*

We will now record two lemmas which will yield an explicit description of the joint law of X and $\hat{X}$, and an intertwining relationship between their respective transition matrices.

Denote by κ_{xy} the number of tableaux of shape x and weight y. (These are the Kostka numbers.)

Lemma 6.3.

$$P(X(n) = y \mid \lambda(m),\ m \le n) = K(\lambda(n), y), \tag{34}$$

where

$$K(x,y) = \frac{p^y}{s_x(p)}\kappa_{xy}. \tag{35}$$

Proof. First note that the σ-algebra generated by $\{\lambda(m),\ m \le n\}$ is precisely the same as the σ-algebra generated by $T(n)$. Thus, the conditional law of $X(n)$, given $\{\lambda(m),\ m \le n\}$, is the same as the conditional law of $X(n)$, given $T(n)$. But this only depends on the shape, $\lambda(n)$, of $T(n)$, and is given by

$$K(x,y) := P(X(n) = y|\ \lambda(n) = x) = \frac{p^y}{s_x(p)}\kappa_{xy},$$

as required. □

We will now show that P and Q are intertwined via the Markov kernel K; that is,

Lemma 6.4. $QK = KP$.

Proof. We have

$$\begin{aligned}(QK)(x,z) &= \sum_{y\in C} Q(x,y)K(y,z)\\ &= \sum_{y\in C} P(x,y)\frac{p^{-y}s_y(p)}{p^{-x}s_x(p)}\frac{p^z}{s_y(p)}\kappa_{yz}\\ &= \sum_i p_i p^x p^{-x-e_i}\frac{p^z}{s_x(p)}\kappa_{x+e_i,z}\\ &= \frac{p^z}{s_x(p)}\sum_i \kappa_{x+e_i,z}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(KP)(x,z) &= \sum_y K(x,y)P(y,z)\\ &= \sum_i K(x,z-e_i)p_i\\ &= \sum_i p_i\frac{p^{z-e_i}}{s_x(p)}\kappa_{x,z-e_i}\\ &= \frac{p^z}{s_x(p)}\sum_i \kappa_{x,z-e_i}.\end{aligned}$$

The statement of the lemma now follows from the identity

$$\sum_i \kappa_{x+e_i,z} = \sum_i \kappa_{x,z-e_i};$$

to see that this holds, observe that, for any $q \in \mathbb{R}_+^k$,

$$\sum_z q^z \sum_i \kappa_{x+e_i,z} = |q|s_x(q) = \sum_z q^z \sum_i \kappa_{x,z-e_i}.$$

□

Corollary 6.5. *If we set* $J(x,y) = K(x^*,y)$, *then*

$$P(X(n) = y \mid \hat{X}(m),\ m \le n,\ \hat{X}(n) = x) = J(x,y) \tag{36}$$

and $\hat{P}J = JP$.

For a discussion on the role of intertwining in the context of Pitman's $2M - X$ theorem (the case $k = 2$), see [38].

It is instructive to note that Theorem 6.1 also follows from Lemmas 6.3 and 6.4, provided we can show that there is a class of functions of the form $K\varphi$, $\varphi : \mathbb{Z}_+^k \to \mathbb{R}$, which separate probability distributions on C. Indeed, by Lemmas 6.3 and 6.4,

$$\begin{aligned}
&E\left[(K\varphi)(\lambda(n+1))|\ \lambda(m), m \leq n\right] \\
&\quad= E\left[E\left[\varphi(X(n+1))|\lambda(n+1)\right]|\lambda(m), m \leq n\right] \\
&\quad= E\left[E\left[\varphi(X(n+1))|\lambda(m), m \leq n+1\right]|\lambda(m), m \leq n\right] \\
&\quad= E\left[\varphi(X(n+1))|\lambda(m), m \leq n\right] \\
&\quad= \sum_y K(\lambda(n), y) E\left[\varphi(X(n+1))|X(n) = y\right] \\
&\quad= \sum_y K(\lambda(n), y) \sum_z P(y,z)\varphi(z) \\
&\quad= [(KP)\varphi](\lambda(n)) \\
&\quad= [(QK)\varphi](\lambda(n)) \\
&\quad= [Q(K\varphi)](\lambda(n)),
\end{aligned}$$

which would imply that λ is a Markov chain with transition matrix Q if the functions $K\varphi$ were determining. To find such a class of functions, we recall that the matrix

$$\{\kappa_{xy}, (x,y) \in C^2\}$$

is invertible (see, for example, [31]). Thus, if we set, for $q \in \mathbb{R}_+^k$,

$$\varphi_q(y) = p^{-y} \sum_{z \in C} \kappa_{yz}^{(-1)} q^z s_z(p) 1_{\{y \in C\}}, \tag{37}$$

we have $(K\varphi_q)(x) = q^x$, and these functions are clearly determining.

By exactly the same arguments as those given in the proof of Theorem 6.1, if $\mu(n)$ denotes the shape of the tableau obtained by applying the Robinson-Schensted algorithm with *row* insertion to the random word $\xi_1 \cdots \xi_n$, we obtain

Theorem 6.6. *μ is a Markov chain on C with transition matrix Q.*

7. Poissonized version

We will now define a continuous version of $G^{(k)}$, and state Poissonized versions of Corollaries 6.2 and 6.5. This is an interesting setting in its own right, since the conditioned walk in this case is closely related to the Charlier ensemble and process (see, for example, [27], [28]), but more importantly it provides a convenient framework in which to apply Donsker's theorem and obtain the Brownian analogue of Corollary 6.2, as was presented in [36] in the case $p = (1/k, \ldots, 1/k)$. Moreover, given the connection we have now made with the Robinson-Schensted algorithm, this continuous path-transformation can also be regarded as a continuous analogue of the Robinson-Schensted algorithm (see section 10 for further remarks in this direction).

Let $D_0(\mathbb{R}_+)$ denote the space of cadlag paths $f : \mathbb{R}_+ \to \mathbb{R}$ with $f(0) = 0$. We will extend the definition of the operations $\triangle$ and $\triangledown$ to a continuous context. For $f, g \in D_0(\mathbb{R}_+)$, define $f \triangle g \in D_0(\mathbb{R}_+)$ and $f \triangledown g \in D_0(\mathbb{R}_+)$ by

$$(f \triangle g)(t) = \inf_{0 \leq s \leq t} [f(s) + g(t) - g(s)] \tag{38}$$

and

$$(f \triangledown g)(t) = \sup_{0 \le s \le t} [f(s) + g(t) - g(s)]. \tag{39}$$

As in the discrete case, these operations are not associative: unless otherwise delineated by parentheses, the default order of operations is from left to right; for example, when we write $f \triangle g \triangle h$, we mean $(f \triangle g) \triangle h$.

Define a sequence of mappings $\Gamma^{(k)} : D_0(\mathbb{R}_+)^k \to D_0(\mathbb{R}_+)^k$ by

$$\Gamma^{(2)}(f, g) = (f \triangle g, g \triangledown f), \tag{40}$$

and, for $k > 2$,

$$\begin{aligned} \Gamma^{(k)}(f_1, \dots, f_k) = (&f_1 \triangle f_2 \triangle \cdots \triangle f_k, \\ &\Gamma^{(k-1)}(f_2 \triangledown f_1, f_3 \triangledown (f_1 \triangle f_2), \dots, f_k \triangledown (f_1 \triangle \cdots \triangle f_{k-1}))). \end{aligned} \tag{41}$$

Let $N = (N_1, \dots, N_k)$ be a continuous-time random walk with generator $Gf(x) = \sum_i \mu_i [f(x+e_i) - f(x)]$. Denote by $\mathbb{R}_x$ the law of N started from x, and by (R_t) the corresponding semigroup. For convenience, we will also denote the corresponding transition kernel by $R_t(x, y)$. Set $p_i = \mu_i/|\mu|$, and denote by $\mathbb{S}_x$ the law of the h_p-transform on W, started at x, and by S_t the corresponding semigroup.

Note that the embedded discrete-time random walk in N has the same law as X. That is, if $\tau_n = \inf\{t \ge 0 : \ |N(t)| = n\}$ and $Y(n) = N(\tau_n)$, then Y is a random walk on $\mathbb{Z}_+^k$ with transition matrix P.

Theorem 7.1. *The law of $M = \Gamma^{(k)}(N)$ under $\mathbb{R}_o$ is the same as the law of N under $\mathbb{S}_o$.*

Moreover, if J is the Markov kernel defined in the previous section, then

Theorem 7.2.

$$\mathbb{R}_o(N(t) = y \mid M(s), \ s \le t) = J(M(t), y), \tag{42}$$

and for $t \ge 0$ we have $S_t J = J R_t$.

8. Brownian motion in a Weyl chamber and random matrices

Let X be a standard Brownian motion in $\mathbb{R}^d$, and let $\mathbb{P}_x$ denote the law of X started at x. Denote the corresponding semigroup and transition kernel by (P_t), and the natural filtration of X by $(\mathcal{F}_t)$.

Recall that

$$W = \{x \in \mathbb{R}^d : \ x_1 \le \cdots \le x_d\}$$

and denote by $\tilde{P}_t$ the semigroup of the process killed at the first exit time

$$T = \inf\{t \ge 0 : \ X(t) \notin W\}.$$

Define $\mathbb{Q}_x$, for $x \in W^\circ$, by

$$\mathbb{Q}_x|_{\mathcal{F}_t} = \frac{h(X(t \wedge T))}{h(x)} \cdot \mathbb{P}_x|_{\mathcal{F}_t},$$

where h is the Vandermonde function $h(x) = \prod_{i<j}(x_j - x_i)$. Denote the corresponding semigroup by Q_t.

The measure

$$\mathbb{Q}_o = \lim_{W^\circ \ni x \to 0} \mathbb{Q}_x$$

is well-defined, and can be interpreted as the law of the eigenvalue-process associated with Hermitian Brownian motion [17], [20]. The law of $X(1)$ under $\mathbb{Q}_o$ is the familiar Gaussian Unitary Ensemble (GUE) of random matrix theory.

In [36] it was shown, by applying Donsker's theorem in the context of Theorem 7.1 with $\mu = (1, \ldots, 1)$, that

Theorem 8.1. *The law of* $\Gamma^{(d)}(X)$ *under* $\mathbb{P}_o$ *is the same as the law of* X *under* $\mathbb{Q}_o$.

In particular,[1] $(X_d \triangledown \cdots \triangledown X_1)(1)$ has the same law as the largest eigenvalue of a $d \times d$ GUE random matrix; this had been observed earlier by Baryshnikov [5] and by Gravner, Tracy and Widom [21]. A similar representation was obtained in [11].

Here we record some additional properties of the process $R = \Gamma^{(d)}(X)$, and its relationship with X, which are inherited, in the same application of Donsker's theorem, from Theorem 7.2.

Theorem 8.2.

$$\mathbb{P}_o(X(t) \in dx \mid R(s),\ s \leq t; R(t) = r) = L(r, dx), \tag{43}$$

where L *is characterised by*

$$\int_{\mathbb{R}^k} e^{\lambda \cdot y} L(x, dy) = \frac{\det(e^{\lambda_i x_j})}{h(\lambda) h(x)} =: c_\lambda(x). \tag{44}$$

Also, for $t \geq 0$ *we have* $Q_t L = L P_t$.

The intertwining $Q_t L = L P_t$ can also be seen as a direct consequence of the Harish-Chandra/Itzykson-Zuber formula [22], [25] for the Laplace transform of the conditional law of the diagonal of a GUE random matrix given its eigenvalues, using the fact that the diagonal of a Hermitian Brownian motion evolves according to the semigroup P_t and the eigenvalues evolve according to the semigroup Q_t. It is also easily verified, using the Karlin-MacGregor formula, that

$$\tilde{P}_t(x, y) = \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) P_t(x, \sigma y).$$

We will now present analogous results for Brownian motion with drift. Fix $\mu \in \mathbb{R}^d$, and denote by $\mathbb{P}_x^{(\mu)}$ the law of Brownian motion in $\mathbb{R}^d$ with drift μ. Denote by $(P_t^{(\mu)})$ the corresponding semigroup and by $(\tilde{P}_t^{(\mu)})$ the semigroup of the process killed at the first exit time T of the Weyl chamber W. Define h_μ by

$$h_\mu(x) = e^{-\mu \cdot x} \left| \det\left(e^{\mu_i x_j}\right) \right|. \tag{45}$$

It is easy to check directly that h_μ is a positive harmonic function for $\tilde{P}_t^{(\mu)}$. Define

$$\mathbb{Q}_x^{(\mu)}\Big|_{\mathcal{F}_t} = \frac{h_\mu(X(t \wedge T))}{h_\mu(x)} \cdot \mathbb{P}_x^{(\mu)}\Big|_{\mathcal{F}_t}.$$

Denote the corresponding semigroup by $Q_t^{(\mu)}$. Recalling the absolute continuity relationship

$$\tilde{P}_t^{(\mu)}(x, y) = e^{\mu \cdot (y - x) - \|\mu\|_2^2 t/2} \tilde{P}_t(x, y), \tag{46}$$

[1] See Remark 2(ii) in section 10 below.

we can write

$$Q_t^{(\mu)}(x,y) = \frac{h_\mu(y)}{h_\mu(x)} \tilde{P}_t^{(\mu)}(x,y) = \frac{\det\left(e^{\mu_i y_j}\right)}{\det\left(e^{\mu_i x_j}\right)} e^{-\|\mu\|_2^2 t/2} \tilde{P}_t(x,y), \tag{47}$$

and we note that this is symmetric in the μ_i.

It is easy to verify that the measure

$$\mathbb{Q}_o^{(\mu)} = \lim_{W^\circ \ni x \to 0} \mathbb{Q}_x^{(\mu)}$$

is well-defined.

Applying Donsker's theorem in the context of Theorem 7.1, as in [36], we obtain

Theorem 8.3. *The law of* $\Gamma^{(d)}(X)$ *under* $\mathbb{P}_o^{(\mu)}$ *is the same as the law of* X *under* $\mathbb{Q}_o^{(\mu)}$.

As in the discrete case, we remark that the law $\mathbb{Q}_o^{(\mu)}$ is symmetric in the drifts μ_i.

We also have, by the same application of Donsker's theorem, the following analogue of Theorem 7.2.

Theorem 8.4.

$$\mathbb{P}_o^{(\mu)}(X(t) \in dx \mid R(s),\ s \le t; R(t) = r) = L^{(\mu)}(r, dx), \tag{48}$$

where

$$L^{(\mu)}(x, dy) = c_\mu(x)^{-1} e^{\mu \cdot y} L(x, dy). \tag{49}$$

Also, for $t \geq 0$,

$$Q_t^{(\mu)} L^{(\mu)} = L^{(\mu)} P_t^{(\mu)}. \tag{50}$$

The intertwining relationship (50) can also be verified directly using $Q_t L = L P_t$ and (46).

For related work on reflecting Brownian motions and non-colliding diffusions see [7], [11], [12], [13], [16], [23], [35], [39] and references therein.

9. An application in queueing theory

In this section, using the connection with the Robinson-Schensted correspondence obtained in section 3, we will write down a formula for the "transient distribution" of a series of $M/M/1$ queues in tandem. There are many papers on this topic for the case of a single queue, where the solution is given in terms of modified Bessel functions; see [2] and references therein. In [3], the case of two queues was considered and a solution obtained, but the techniques used there do not seem to extend easily to higher dimensions.

Consider a series of $M/M/1$ queues in tandem, k in number, driven by Poisson processes $N_1, \ldots, N_k$ with respective intensities $\mu_1, \ldots, \mu_k$. The first queue has infinitely many customers, and the remaining queues are initially empty. At every point of N_i, there is a service at the i^{th} queue and, provided that queue is not empty, a customer departs and joins the $(i+1)^{th}$ queue (or leaves the system if $i = k$).

Denote by $D(t) = (D_1(t), \ldots, D_k(t))$ the respective numbers of customers to depart from each queue up to time t. Note that, since there are always infinitely many customers in the first queue, D_1 is a Poisson process; we can thus ignore the first queue, think of the second queue as the first in a series of $k-1$ queues, and think of D_1 as the arrival process at the first of these $k-1$ queues in the series. This is a more conventional setup in queueing theory. The state of the system is described by the queue lengths

$$Q_1 = D_1 - D_2, \ \dots \ , \ Q_{k-1} = D_{k-1} - D_k. \tag{51}$$

We will write down a formula for the law of $D(t)$ which, in turn, yields the law of $Q(t) = (Q_1(t), \dots, Q_{k-1}(t))$.

Without loss of generality, we can assume that $|\mu| = 1$. The de-Poissonized version of this problem is to consider the usual random walk X, with $p = \mu$, and consider the law of $\delta(n) = (D^{(k)}(X))(n)$. But we know this law, from sections 3 and 6. It is the law of $\beta(S(n))$, where $S(n)$ of the random semistandard tableau obtained when one applies the Robinson-Schensted algorithm with column-insertion to the random word $\xi_1 \cdots \xi_k$ and $\beta_i(\tau)$ denotes the number of i's in the i^{th} row of a tableau τ.

Thus,

$$P(D(t) = d) = e^{-t} \sum_{n \geq 0} \frac{t^n}{n!} P(\delta(n) = d), \tag{52}$$

where

$$P(\delta(n) = d) = \sum_{l \geq d, l \vdash n} \sum_{\mathrm{sh}\, \tau = l} p^\tau f_l 1_{\{\beta(\tau) = d\}}. \tag{53}$$

This formula is complicated in general, but simplifies in certain cases.

Consider the case $k = 2$. In this case, we have only one summand:

$$P(\delta(n) = d) = p_1^{d_1} p_2^{n-d_1} f_{(n-d_2, d_2)}. \tag{54}$$

By the hook-length formula (see, for example, [18]), for $n \geq d_1 + d_2$,

$$f_{(n-d_2,d_2)} = n! \frac{n - 2d_2 + 1}{(n - d_2 + 1)! d_2!}.$$

Thus, recalling that $p = \mu$,

$$P(D(t) = d) = e^{-t} \sum_{n \geq d_1 + d_2} t^n \mu_1^{d_1} \mu_2^{n-d_1} \frac{n - 2d_2 + 1}{(n - d_2 + 1)! d_2!}. \tag{55}$$

It follows that

$$P(Q(t) = q) = (\mu_1/\mu_2)^q e^{-t} \sum_{m \geq q} (m+1)(\mu_2 t)^m I_{m+1}(2\sqrt{\mu_1 \mu_2} t). \tag{56}$$

Let $s_{l/d}$ denote the Schur polynomial associated with the skew-tableau l/d (see, for example, [18]). We will use the following formula:

$$\sum_l s_{l/d}(x) \frac{t^{|l|} f_l}{|l|!} = e^{|x|t} \frac{t^{|d|} f_d}{|d|!}. \tag{57}$$

This follows from the identity

$$\sum_l s_{l/d}(x) s_l(y) = s_d(y) \prod_{i,j} (1 - x_i y_j)^{-1} \tag{58}$$

(this is a variant of Cauchy's identity; see, for example, [31, pp. 62–70]) and the fact that

$$\lim_{n\to\infty} s_l\left(\frac{t}{n}.1^n\right) = \frac{t^{|l|}f_l}{|l|!}. \tag{59}$$

In the case $k = 3$, if $p_2 = p_3$, we have

$$\begin{aligned} P(D_1(t) = d_1, D_2 \geq d_2, D_3(t) = d_3) &= e^{-t}p^d \sum_l s_{l/d}(p_2, p_3)\frac{t^{|l|}f_l}{|l|!} \\ &= e^{-p_1 t}p^d \frac{t^{|d|}f_d}{|d|!}. \end{aligned} \tag{60}$$

It would be interesting to compare (60) with the explicit formulas obtained in [3] for this case.

In the general case, we can simplify the formula (52) if d is constant. Suppose $d_i = m$ for all i. Then, using (57) and the hook-length formula,

$$P(D(t) = d) = e^{-t}p^d \sum_l s_{l/d}(p_2, p_3, \ldots, p_k)\frac{t^{|l|}f_l}{|l|!} \tag{61}$$

$$= e^{-p_1 t}p^d \frac{t^{|d|}f_d}{|d|!} \tag{62}$$

$$= e^{-p_1 t}(t^k p_1 p_2 \cdots p_k)^m \prod_{i\leq k} \frac{\Gamma(i+m)}{\Gamma(i)}. \tag{63}$$

It follows that

$$P(Q_1(t) = Q_2(t) = \cdots = Q_{k-1}(t) = 0) = e^{-p_1 t}H(t^k p_1 p_2 \cdots p_k), \tag{64}$$

where

$$H(s) = \sum_{m\geq 0} s^m \prod_{i\leq k} \frac{\Gamma(i)}{\Gamma(i+m)}. \tag{65}$$

Finally we remark that, by Theorem 6.1 and the symmetry of the Schur polynomials, the law of the process D_k is symmetric in the parameters $p_1, \ldots, p_k$.

10. Concluding remarks

1. *The Krawchouk process:* A binomial version of Theorem 1.1 was presented in [28]. This states that, if X is a random walk in Z_+^k with transition matrix

$$P(x,y) = Cp^{y-x}1_{y-x\in\{0,1\}^k}$$

(C is a normalising constant), then, assuming $p_1 < \cdots < p_k$ and extending slightly the domain of $G^{(k)}$, we see that $G^{(k)}(X)$ has the same law as that of X conditioned to stay forever in W. In this case, a similar connection can be made with the dual Robinson-Schensted-Knuth (RSK) correspondence for zero-one matrices, and analogues of all the main results of section 6 can be obtained similarly. The Schur polynomials again play an important role. See [34] for details.

2. *Properties of* $\Gamma^{(k)}$*:* The following continuous analogues of Lemmas 2.1–2.4 can be readily verified. Denote by $C_0(\mathbb{R}_+, \mathbb{R}^k)$ the set of continuous functions $f : \mathbb{R}_+ \to \mathbb{R}^k$ with $f(0) = o$. For $f \in C_0(\mathbb{R}_+, \mathbb{R}^k)$, where $k \geq 2$,

(i) $|\Gamma^{(k)}(f)| = |f|$,

(ii) $\Gamma_k^{(k)}(f) = f_k \triangledown \cdots \triangledown f_1$,
(iii) $f(t) = [\Gamma^{(k)}(f)](t) + \Phi^{(k)}([\Gamma^{(k)}(f)](t,u),\ u \geq t)$, where $\Phi^{(k)}$ is defined on a suitable domain as the continuous analogue of $F^{(k)}$.

In this continuous setting, the identity (ii) can be verified directly using the "sup-integration by parts" formula

$$\sup_{0<s<t}\left\{\sup_{0<r<s} u(r) + v(s)\right\} \bigvee \sup_{0<s<t}\left\{u(s) + \sup_{0<r<s} v(r)\right\} = \sup_{0<s<t} u(s) + \sup_{0<s<t} v(s),$$

for $u, v \in C_0(\mathbb{R}_+, \mathbb{R}^k)$. This is a "max-plus" analogue of the usual integration by parts formula and is easily verified by the method of Laplace.

3. *A continuous Robinson-Schensted algorithm:* Given the connection we have made between the path-transformations $G^{(k)}$ and the Robinson-Schensted algorithm, the mappings $\Gamma^{(k)}$ can be used to define a continuous version of the Robinson-Schensted algorithm. More precisely, let $\mathbb{C}_{GC}$ denote the *Gelfand-Cetlin cone*, which consists of triangular arrays of real numbers

$$(x_j^{(i)},\ 1 \leq i \leq k,\ 1 \leq j \leq i) \tag{66}$$

satisfying $x_j^{(i)} \geq x_j^{(i-1)} \geq x_{j+1}^{(i)}$, for all i, j. Points in the Gelfand-Cetlin cone can be regarded as continuous analogues of semistandard tableaux. The continuous analogue of a word is a continuous function $f : [0,1] \to \mathbb{R}^k$ with $f(0) = o$: denote the set of these functions by $C_0([0,1], \mathbb{R}^k)$. Define a map $\phi : C_0([0,1], \mathbb{R}^k) \to \mathbb{C}_{GC}$ as follows. For convenience, let $\Gamma^{(1)}$ be the identity transformation. If we set $x = \phi(f_1, \dots, f_k)$, then, for each $1 \leq i \leq k$,

$$x^{(i)} = ([\Gamma_i^{(i)}(f_1, \dots, f_i)](1), \dots, [\Gamma_1^{(i)}(f_1, \dots, f_i)](1)). \tag{67}$$

The continuous analogue of the corresponding "recording tableau" is the path

$$\rho(f) = \{[\Gamma^{(k)}(f)](t),\ 0 \leq t \leq 1\} \in C_0([0,1], W). \tag{68}$$

By analogy with the discrete Robinson-Schensted algorithm, the function f can be uniquely recovered from the pair $\phi(f)$ and $\rho(f)$. A more detailed discussion on the properties of this continuous Robinson-Schensted algorithm will be presented elsewhere.

4. *GUE minors:* Let A be a $k \times k$ GUE random matrix, and denote the eigenvalues of the i^{th} minor $(A_{lm},\ l, m \leq i)$ by $\lambda_1^{(i)} \geq \cdots \geq \lambda_i^{(i)}$, for $i \leq k$. In the above context, Baryshnikov [5] showed that, if $(B(t),\ 0 \leq t \leq 1)$ is a standard Brownian motion in $\mathbb{R}^k$, then the random vector

$$(\phi_1^{(1)}(B), \dots, \phi_1^{(k)}(B))$$

has the same law as

$$(\lambda_1^{(1)}, \dots, \lambda_1^{(k)}).$$

In [5], Donsker's theorem is applied in the context of a random semistandard tableau with the same law as $T(n)$ of section 6 in the homogeneous case $p_1 = \cdots = p_k$. We can thus extend Baryshnikov's arguments using the representation for the Robinson-Schensted algorithm given in this paper and the continuity of the mappings $\Gamma^{(i)}$, $i \leq k$, to see that, in fact, $\phi(B)$ has the same law as

$$(\lambda^{(1)}, \dots, \lambda^{(k)}).$$

However, it is easy to see,[2] by considering the case $k=2$, that these identities do not extend to the process level (that is, with ϕ defined simultaneously on intervals $[0,t]$ instead of just $[0,1]$ and "GUE" replaced by "Hermitian Brownian motion").

5. *Related topics:* The intertwining of Lemma 6.4 is closely related to the work of Biane on quantum random walks [8], [9], [10]. The Robinson-Schensted correspondence is of fundamental importance to the representation theory of S_n and $GL(n)$. Related topics in representation theory (which are certainly connected to results presented in this paper) include Littelmann's path model for the finite-dimensional representations of $GL(n)$ (see, for example, [30]), crystal bases, and representations of quantum groups (see, for example, [14], [29]).

Appendix

Proof of Lemma 2.1. The first identity is trivial:

$$\begin{aligned} (y \triangledown x)(n) &= \max_{0\le m\le n}[y(m)+x(n)-x(m)] \\ &= x(n)+y(n)+\max_{0\le m\le n}[y(m)-y(n)-x(m)] \\ &= x(n)+y(n)-(x \vartriangle y)(n). \end{aligned}$$

If we set $z=y-x$, and $s(n)=\max_{0\le m\le n} z(n)$, then the second identity is equivalent to the well-known fact that

$$s(n)=\min_{l\le n}[2s(l)-z(l)].$$

□

Proof of Lemma 2.2. Fix $x\in\Lambda_k$, and write $G^{(k)}=G^{(k)}(x)$ unless otherwise indicated (similarly for $D^{(k)}$ and $T^{(k)}$). We will first show that $|G^{(k)}|=|x|$. We will prove this by induction on k. The case $k=2$ is given by Lemma 2.1. Assume the induction hypothesis for $k-1$. We recall from the definitions that

$$G^{(k)}=\left(D_k^{(k)},G^{(k-1)}\left(T^{(k)}\right)\right). \tag{69}$$

By the induction hypothesis,

$$|G^{(k)}|=D_k^{(k)}+|T^{(k)}|. \tag{70}$$

Recall that, for $i\ge 2$,

$$D_i^{(k)}=D_{i-1}^{(k)} \vartriangle x_i \tag{71}$$

and

$$T_{i-1}^{(k)}=x_i \triangledown D_{i-1}^{(k)}. \tag{72}$$

Thus, by Lemma 2.1,

$$x_i=D_i^{(k)}-D_{i-1}^{(k)}+T_{i-1}^{(k)} \tag{73}$$

for each $i\ge 2$; summing this over i and recalling that $D_1^{(k)}=x_1$ yields

$$|x|=D_k^{(k)}+|T^{(k)}|; \tag{74}$$

so we are done.

[2] Bougerol and Jeulin, private communication

We will now show that

$$x(n) = G^{(k)}(n) + F^{(k)}\left(G^{(k)}(l) - G^{(k)}(n),\ l \geq n\right), \tag{75}$$

for some function $F^{(k)}$ to be defined. Again we will prove this by induction on k, and note that for $k = 2$, this is given by Lemma 2.1.

A recursive definition of $F^{(k)}$ will be implicit in the induction argument. Recall that

$$G^{(k)} = \left(D_k^{(k)}, G^{(k-1)}\left(T^{(k)}\right)\right). \tag{76}$$

Assuming the induction hypothesis for $k-1$, we have, for $i \geq 2$,

$$T_{i-1}^{(k)}(n) = G_i^{(k)}(n) + F^{(k-1)}\left((G_2^{(k)}, \dots, G_k^{(k)})(n,l)\right). \tag{77}$$

Thus, for $i \geq 2$, using (73) and the fact that

$$D_i^{(k)}(n) - D_{i-1}^{(k)}(n) = \max_{l \geq n}[D_i^{(k)}(n,l) - T_{i-1}^{(k)}(n,l)],$$

we have

$$x_i(n) = G_i^{(k)}(n) + J_i^{(k)}\left((D_i^{(k)}, G_2^{(k)}, \dots, G_k^{(k)})(n,l)\right), \tag{78}$$

where $J_i^{(k)}$ is defined on a suitable domain. It is important to note here that $J_i^{(k)}$ does not depend on n.

In this way, recalling that $D_k^{(k)} = G_1^{(k)}$, we obtain

$$x_k(n) = G_k^{(k)}(n) + F_k^{(k)}\left(G^{(k)}(l) - G^{(k)}(n),\ l \geq n\right), \tag{79}$$

where the function $F_k^{(k)}$ is implicitly defined by this identity (on a suitable domain) and does not depend on n. Observe that we can also recover the sequence of future increments $x_k(l) - x_k(n)$ as a function, which does not depend on n, of the sequence $\{G^{(k)}(l) - G^{(k)}(n),\ l \geq n\}$.

We can now recover the values $x_{k-1}(n)$, $x_{k-2}(n)$, and so on, as follows. By equations (73) with $i = k$,

$$D_{k-1}^{(k)}(n) = G_1^{(k)}(n) - x_k(n) + T_{k-1}^{(k)}(n). \tag{80}$$

It follows that the sequence $\{D_{k-1}^{(k)}(n,l),\ l \geq n\}$ is a function, which does not depend on n, of the sequence $\{G^{(k)}(l) - G^{(k)}(n),\ l \geq n\}$. Combining this with (78), we see that

$$x_{k-1}(n) = G_{k-1}^{(k)}(n) + F_{k-1}^{(k)}\left(G^{(k)}(l) - G^{(k)}(n),\ l \geq n\right), \tag{81}$$

where $F_{k-1}^{(k)}$ is implicitly defined by this identity (on a suitable domain) and does not depend on n. Similarly, we can recover the sequence of future increments $x_k(l) - x_k(n)$ as a function, which does not depend on n, of the sequence $\{G^{(k)}(l) - G^{(k)}(n),\ l \geq n\}$, and so on. Finally, $x_1(n)$ is obtained using $|x| = |G^{(k)}|$. □

Proof of Lemma 2.3. We want to show that, for $(a,b,c) \in \Lambda_3$,

$$a \triangledown (c \vartriangle b) \triangledown (b \triangledown c) = a \triangledown b \triangledown c, \tag{82}$$

and for $(w,x,y) \in \Lambda_3$,

$$w \vartriangle (y \triangledown x) \vartriangle (x \vartriangle y) = w \vartriangle x \vartriangle y. \tag{83}$$

First note that these identities are equivalent. To see this, set $a(n) = n - w(n)$, $b(n) = n - x(n)$ and $c(n) = n - y(n)$. Then plug these into (82) to obtain (83). We will therefore restrict our attention to the identity (83).

Let $d = x \triangle y$, $t = y \triangledown x$, $q = x - d$ and $u = y - d$. Then (83) becomes

$$w \triangle (x+u) \triangle (y-u) = w \triangle x \triangle y. \tag{84}$$

That is, the output of a series of queues in tandem driven by $(w, x+u, y-u)$ is the same as that of the series driven by (w, x, y). Set

$$\begin{aligned} d_1 &= w \triangle x, \\ d_2 &= w \triangle x \triangle y, \\ \tilde{d}_1 &= w \triangle (x+u), \\ \tilde{d}_2 &= w \triangle (x+u) \triangle (y-u), \end{aligned}$$

and

$$\begin{aligned} q_1 &= w - d_1, \\ q_2 &= d_1 - d_2, \\ \tilde{q}_1 &= w - \tilde{d}_1, \\ \tilde{q}_2 &= \tilde{d}_1 - \tilde{d}_2. \end{aligned}$$

We want to show that $d_2 = \tilde{d}_2$. From the above definitions, this is equivalent to showing that

$$q_1(n) + q_2(n) = \tilde{q}_1(n) + \tilde{q}_2(n)$$

for all $n \geq 0$. We will prove this by induction on n.

The induction hypothesis H is:

$q_1 + q_2 = \tilde{q}_1 + \tilde{q}_2$, and *either*

(i) $\tilde{q}_2 - q_2 \geq 0$ and $q - q_2 = 0$, *or*

(ii) $\tilde{q}_2 - q_2 = 0$ and $q - q_2 \geq 0$.

When $n = 0$ we have $q = q_1 = q_2 = \tilde{q}_1 = \tilde{q}_2 = 0$, and the induction hypothesis is trivially satisfied. Assume the induction hypothesis holds at time $n-1$. Note that $(w, x, y-u, u) \in \Lambda_4$; that is, only one of these quantities, if any, can increase by one at time n. We will consider the following five cases, which are exhaustive and mutually exclusive, separately.

(a) $(w, x, y-u, u)(n) = (w, x, y-u, u)(n-1)$,
(b) $w(n) - w(n-1) = 1$,
(c) $x(n) - x(n-1) = 1$,
(d) $(y-u)(n) - (y-u)(n-1) = 1$,
(e) $u(n) - u(n-1) = 1$.

Case (a): $(w, x, y-u, u)(n) = (w, x, y-u, u)(n-1)$. In this case, nothing changes, and so H is preserved.

Case (b): $w(n) - w(n-1) = 1$. In this case, $q_1(n) = q_1(n-1) + 1$ and $\tilde{q}_1(n) = \tilde{q}_1(n-1) + 1$, the other quantities remain unchanged, and H is preserved.

Case (c): $x(n) - x(n-1) = 1$. Then $q(n) = q(n-1) + 1$.

Suppose $q_1(n-1) > \tilde{q}_1(n-1) > 0$. Then $q_1(n) = q_1(n-1) - 1$ and $q_2(n) = q_2(n-1) + 1$. Thus, $q - q_2$ and $q_1 + q_2$ do not change. Also, $\tilde{q}_1(n) = \tilde{q}_1(n-1) - 1$ and $\tilde{q}_2(n) = \tilde{q}_2(n-1) + 1$. Thus, $\tilde{q}_2 - q_2$ and $\tilde{q}_1 + \tilde{q}_2$ do not change either; so we still have $q_1 + q_2 = \tilde{q}_1 + \tilde{q}_2$, and H is preserved.

Now suppose $q_1(n-1) > \tilde{q}_1(n-1) = 0$. Note that this implies $\tilde{q}_2(n-1) - q_2(n-1) > 0$, so that we are initially in case (i) of the induction hypothesis. In this case, $q_1(n) = q_1(n-1) - 1$ and $q_2(n) = q_2(n-1) + 1$, but $\tilde{q}_1$ and $\tilde{q}_2$ do not change. Thus, $q - q_2$, $q_1 + q_2$ and $\tilde{q}_1 + \tilde{q}_2$ do not change. The quantity $\tilde{q}_2 - q_2$ decreases by one, but remains nonnegative; so we remain in case (i), and H is preserved.

Finally, if $q_1(n-1) = \tilde{q}_1(n-1) = 0$, then we are initially in case (ii) of H. There is no change to $q_1, q_2, \tilde{q}_1$ or $\tilde{q}_2$, but q inceases by one, and so we remain in case (ii), and H is preserved.

Case (d): $(y-u)(n) - (y-u)(n-1) = 1$. In this case, $q(n-1) > 0$ and q decreases by one. The values of q_1 and $\tilde{q}_1$ do not change.

If we are in case (i) at time $n-1$, then $\tilde{q}_2(n-1) \geq q_2(n-1) > 0$, and so both q_2 and $\tilde{q}_2$ also decrease by one; thus, we remain in case (i), and H is preserved.

If we are in case (ii) at time $n-1$, then $\tilde{q}_2(n-1) = q_2(n-1)$, and either q_2 and $\tilde{q}_2$ both decrease by one or both remain unchanged. Either way, we remain in case (ii), and H is preserved.

Case (e): $u(n) - u(n-1) = 1$. Then $q(n-1) = q_2(n-1) = 0$, and we are initially in case (i) of H. The values of q, q_1 and q_2 will not change. If $\tilde{q}_1 > 0$, then $\tilde{q}_1$ decreases by one and $\tilde{q}_2$ increases by one; otherwise, $\tilde{q}_1$ and $\tilde{q}_2$ do not change. Either way, we remain in case (i), and H is preserved. □

Proof of Lemma 2.4. We will prove this by induction on k. It is certainly true for $k = 2$, from the definition of $G^{(2)}$. Recall the definition of $G^{(k)}$,

$$G^{(k)} = \left(D_k^{(k)}, G^{(k-1)}\left(T^{(k)}\right)\right). \tag{85}$$

By the induction hypothesis, that the lemma is true for $G^{(k-1)}$, we have

$$G_k^{(k)} = T_{k-1}^{(k)} \triangledown T_{k-2}^{(k)} \triangledown \cdots \triangledown T_1^{(k)}. \tag{86}$$

We will now repeatedly apply Lemma 2.3:

$$\begin{aligned} T_{k-1}^{(k)} \triangledown T_{k-2}^{(k)} &= x_k \triangledown D_{k-1}^{(k)} \triangledown T_{k-2}^{(k)} \\ &= x_k \triangledown (D_{k-2}^{(k)} \vartriangle x_{k-1}) \triangledown (x_{k-1} \triangledown D_{k-2}^{(k)}) \\ &= x_k \triangledown x_{k-1} \triangledown D_{k-2}^{(k)}. \end{aligned}$$

Similarly,

$$\begin{aligned} x_k \triangledown x_{k-1} \triangledown D_{k-2}^{(k)} \triangledown T_{k-3}^{(k)} &= x_k \triangledown x_{k-1} \triangledown (D_{k-3}^{(k)} \vartriangle x_{k-2}) \triangledown (x_{k-2} \triangledown D_{k-3}^{(k)}) \\ &= x_k \triangledown x_{k-1} \triangledown x_{k-2} \triangledown \ D_{k-3}^{(k)}, \end{aligned}$$

and so on. □

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