

Stochastic Bäcklund Transformations

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Dedicated to the memory of Marc Yor

Abstract How does one introduce randomness into a classical dynamical system in order to produce something which is related to the ‘corresponding’ quantum system? We consider this question from a probabilistic point of view, in the context of some integrable Hamiltonian systems.

1 Introduction

Let $\mu \geq 1/2$ and consider the evolution $\dot{x} = \mu/x$ on the positive half-line. Then $\ddot{x} = -\mu^2/x^3$, which is the equation of motion for the rational Calogero-Moser system with Hamiltonian

$$\frac{1}{2}p^2 - \frac{\mu^2}{2x^2}.$$

If we add noise, that is, if we consider the stochastic differential equation

$$dX = dB + \frac{\mu}{X}dt,$$

where B is a standard one-dimensional Brownian motion, then X is a diffusion process on the positive half-line with infinitesimal generator

$$L = \frac{1}{2}\partial_x^2 + \frac{\mu}{x}\partial_x.$$

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The assumption $\mu \geq 1/2$ ensures that X never hits zero. The operator L is related to the quantum Calogero-Moser Hamiltonian

$$H = \frac{1}{2} \partial_x^2 - \frac{\mu(\mu - 1)}{2x^2},$$

via the ground state transform

$$L = \psi(x)^{-1} H \psi(x),$$

where $\psi(x) = x^\mu$. Ignoring for the moment the discrepancy between the coupling constants $\mu^2/2$ and $\mu(\mu - 1)/2$, this provides a very simple example of a classical Hamiltonian system which has the property that if we add noise in a suitable way we obtain a diffusion process whose infinitesimal generator is simply related to the corresponding quantum system.

Appealing as it is, this example is quite unique and, in fact, somewhat misleading. The only constant of motion is the Hamiltonian itself and, as $\dot{x} = p$, the evolution $\dot{x} = \mu/x$ necessarily has $p^2/2 - \mu^2/2x^2 = 0$. It is not clear how to extend this construction—together with its stochastic counterpart—to allow for other values. In fact, there is a kind of ‘explanation’ for this limitation which will come later.

In the papers [21, 22] a certain probabilistic relation between the classical and quantum Toda lattice was observed. This relation can be loosely described as follows: starting with a particular construction of the classical flow on a given sub-Lagrangian manifold, *adding white noise to the constants of motion* yields a diffusion process whose infinitesimal generator is simply related to the corresponding quantum system.

As we shall see, this relation extends naturally to some other integrable many-body systems, specifically rational and hyperbolic Calogero-Moser systems. The basic construction can be formulated in terms of *kernel functions* and *Bäcklund transformations*. For more background on the (interrelated) role of kernel functions and Bäcklund transformations in integrable systems see, for example, [9, 16, 24, 27] and references therein. In the present paper, to illustrate the main ideas, we will focus on rank-one (two particle) systems although most of the constructions extend naturally to higher rank systems.

The examples we consider are of course very special, having the property that there are kernel functions which unite the classical and quantum systems through a kind of exact stationary phase property. Nevertheless, they should provide a useful benchmark for exploring similar relations for other Hamiltonian systems.

The outline of the paper is as follows. In the next section, we illustrate the basic construction of [21, 22] in the context of the rank one Toda lattice. In this setting it is closely related to earlier results of Matsumoto and Yor [17] and Baudoin [1]. In Sects. 3–5, we give analogous constructions for the rational and hyperbolic Calogero-Moser systems. As we shall see, the above example should in fact be seen as a particular degeneration of a more general construction for the hyperbolic Calogero-Moser system, based on the kernel functions of Hallnäs and

Ruijsenaars [9, 10]. In Sect. 6, we conclude with some remarks on how the solution to the Kardar-Parisi-Zhang equation can also be interpreted from this point of view.

The following notation will be used throughout. If E is a topological space, we denote by $B(E)$ the set of Borel measurable functions on E , by $C_b(E)$ the set of bounded continuous functions on E and by $\mathcal{P}(E)$ the set of Borel probability measures on E . If E is an open subset of \mathbb{R}^n , we denote by $C_c^2(E)$ the set of continuously twice differentiable, compactly supported, functions on E .

2 The Toda Lattice

For the rank-one Toda lattice we consider the kernel function

$$K(x, u) = \exp(-e^{-x} \cosh u),$$

and note that K satisfies

$$(\partial_x \ln K)^2 - (\partial_u \ln K)^2 = e^{-2x}, \tag{1}$$

and

$$\partial_x^2 \ln K - \partial_u^2 \ln K = 0. \tag{2}$$

The corresponding Bäcklund transformation

$$\dot{u} = -\partial_u \ln K = e^{-x} \sinh u, \quad \dot{x} = \partial_x \ln K = e^{-x} \cosh u \tag{3}$$

has the property that, if (3) holds, then x satisfies the equations of motion of the Toda system with Hamiltonian

$$\frac{1}{2}p^2 - \frac{1}{2}e^{-2x},$$

and $\dot{u} = \lambda$ is a conserved quantity for the coupled system. Indeed, differentiating (1) with respect to x yields

$$\ddot{x} = \partial_x^2 \ln K \partial_x \ln K - \partial_u \partial_x \ln K \partial_u \ln K = -e^{-2x}, \tag{4}$$

and differentiating (1) with respect to u gives

$$\ddot{u} = \partial_u^2 \ln K \partial_u \ln K - \partial_x \partial_u \ln K \partial_x \ln K = 0. \tag{5}$$

It also follows from (1) and (3) that λ is an eigenvalue of the Lax matrix

$$\begin{pmatrix} p & e^{-x} \\ -e^{-x} & -p \end{pmatrix}.$$

Now the equation $\dot{u} = \lambda$ is equivalent to the critical point equation $\partial_u \ln K_\lambda = 0$, where $K_\lambda = e^{\lambda u} K$. Using this equation, namely

$$\sinh u = \lambda e^x, \quad (6)$$

we can rewrite the evolution equation (3) as

$$\dot{u} = \lambda, \quad \dot{x} = \lambda + e^{-u-x} = (\partial_x + \partial_u) \ln K_\lambda. \quad (7)$$

We note that (6) has a unique solution $u_\lambda(x) = \sinh^{-1}(\lambda e^x)$ for any $\lambda, x \in \mathbb{R}$. The relation (6) is stable under the new evolution equations (7), and is now required to be in force in order to guarantee that (x, p) evolves according to the Toda flow on the iso-spectral manifold corresponding to λ . Given any $\lambda \in \mathbb{R}$, the evolution equations (7) are well-posed on the corresponding iso-spectral manifold (defined in these coordinates by the relation (6)) in the sense that they admit a unique semi-global solution. For any $\lambda \in \mathbb{R}$ and initial condition $x(0) = x_0$, the solution is given explicitly for all $t \geq 0$ by

$$u(t) = u_\lambda(x_0) + \lambda t, \quad x(t) = \begin{cases} \ln\left(\frac{1}{\lambda} \sinh u(t)\right) & \lambda \neq 0 \\ \ln(e^{x_0} + t) & \lambda = 0. \end{cases}$$

The evolution equations (7) provide the correct framework into which we can introduce noise with the desired outcome.

Let $H = (\partial_x^2 - e^{-2x})/2$, and write $H_\lambda = H - \lambda^2/2$. Combining (1) and (2) gives the intertwining relation

$$H_\lambda K_\lambda = \left(\frac{1}{2} \partial_u^2 - \lambda \partial_u\right) K_\lambda. \quad (8)$$

It follows, using the Leibnitz rule, that

$$\psi_\lambda(x) = \int_{-\infty}^{\infty} K_\lambda(x, u) du$$

is an eigenfunction of H with eigenvalue $\lambda^2/2$. We note that $\psi_\lambda(x) = 2K_\lambda(e^{-x})$, where $K_\nu(z)$ is the modified Bessel function of the second kind, also known as Macdonald's function.

Remark 1 The intertwining relation (8) and associated integral formula for the eigenfunctions can be seen as a special case of those obtained by Gerasimov, Kharchev, Lebedev and Oblezin [8] for the n -particle open Toda chain. The above Bäcklund transformation is a special case of the one given by Wojciechowski [29] which, as remarked in that paper, is closely related to a construction of Kac and Van Moerbeke [11] for the periodic Toda chain. It can also be seen as a particular degeneration of the Bäcklund transformation for the infinite particle system given in Toda's monograph [28].

Consider the integral operator defined, for suitable $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, by

$$\tilde{K}_\lambda f(x) = \int_{-\infty}^{\infty} K_\lambda(x, u) f(x, u) du,$$

and the differential operator, defined on $\mathcal{D}(A_\lambda) = C_c^2(\mathbb{R}^2)$, by

$$A_\lambda = \frac{1}{2} \partial_x^2 + \frac{1}{2} \partial_u^2 + \partial_x \partial_u + \lambda \partial_u + (\lambda + e^{-u-x}) \partial_x.$$

Proposition 1 For $f \in \mathcal{D}(A_\lambda)$,

$$H_\lambda \tilde{K}_\lambda f = \tilde{K}_\lambda A_\lambda f. \quad (9)$$

Proof This follows from the intertwining relation (8). Recall that

$$(\partial_x + \partial_u) \ln K_\lambda = \lambda + e^{-u-x}.$$

By Leibnitz' rule and integration by parts,

$$\begin{aligned} H_\lambda \tilde{K}_\lambda f(x) &= H_\lambda \int_{-\infty}^{\infty} K_\lambda f du \\ &= \int_{-\infty}^{\infty} \left[(H_\lambda K_\lambda) f + (\partial_x K_\lambda) \partial_x f + K_\lambda \frac{1}{2} \partial_x^2 f \right] du \\ &= \int_{-\infty}^{\infty} \left[\left(\frac{1}{2} \partial_u^2 K_\lambda - \lambda \partial_u K_\lambda \right) f + K_\lambda (\partial_x \ln K_\lambda) \partial_x f + K_\lambda \frac{1}{2} \partial_x^2 f \right] du \\ &= \int_{-\infty}^{\infty} \left[K_\lambda \left(\frac{1}{2} \partial_u^2 f + \lambda \partial_u f \right) + K_\lambda ((\partial_x + \partial_u) \ln K_\lambda) \partial_x f - (\partial_u K_\lambda) \partial_x f + K_\lambda \frac{1}{2} \partial_x^2 f \right] du \\ &= \int_{-\infty}^{\infty} \left[K_\lambda \left(\frac{1}{2} \partial_u^2 f + \lambda \partial_u f \right) + K_\lambda ((\partial_x + \partial_u) \ln K_\lambda) \partial_x f + K_\lambda \partial_u \partial_x f + K_\lambda \frac{1}{2} \partial_x^2 f \right] du \\ &= K_\lambda A_\lambda f, \end{aligned}$$

as required. \square

Now, if $\lambda \in \mathbb{R}$, the intertwining relation (9) has a *probabilistic* meaning, which we will soon make precise. It implies that there is a two-dimensional diffusion process, characterized by the differential operator A_λ , which has the property that, with particular initial condition specified by the kernel K_λ , its projection onto the x -coordinate is a diffusion process in \mathbb{R} which is characterised by a renormalisation

of the operator H_λ . Moreover, the two-dimensional diffusion process characterized by A_λ is precisely the Bäcklund transformation, in the form of (7), with white noise added to the constant of motion λ . We will now make this statement precise.

Suppose $\lambda \in \mathbb{R}$, let B be a standard one-dimensional Brownian motion and consider the coupled stochastic differential equations obtained by adding white noise to λ in (7), that is

$$dU = dB + \lambda dt, \quad dX = dU + e^{-U-X} dt. \quad (10)$$

This can be solved explicitly: for any initial condition (X_0, U_0) ,

$$U_t = U_0 + B_t + \lambda t, \quad X_t = U_t + \ln \left(e^{X_0 - U_0} + \int_0^t e^{-2U_s} ds \right). \quad (11)$$

As the function $(x, u) \mapsto (\lambda + e^{-u-x}, \lambda)$ is locally Lipschitz, it follows that, for any initial condition, (11) is the unique solution to (10). Moreover, it is a diffusion process in \mathbb{R}^2 with infinitesimal generator A_λ and the martingale problem for (A_λ, ν) is well-posed for any $\nu \in \mathcal{P}(\mathbb{R}^2)$. For more background on the relation between stochastic differential equations and martingale problems see, for example, [6, 15].

Next we consider the diffusion process on \mathbb{R} with infinitesimal generator

$$L_\lambda = \psi_\lambda(x)^{-1} H_\lambda \psi_\lambda(x) = \frac{1}{2} \partial_x^2 + \partial_x \ln \psi_\lambda(x) \cdot \partial_x.$$

This process was introduced by Matsumoto and Yor [17]. Observe that the drift

$$\partial_x \ln \psi_\lambda(x) = e^{-x} \frac{K_{\lambda+1}(e^{-x})}{K_\lambda(e^{-x})},$$

is locally Lipschitz, behaves like e^{-x} at $-\infty$ and vanishes at $+\infty$. It follows that $-\infty$ is an entrance boundary, $+\infty$ is a natural boundary and, for any $\rho \in \mathcal{P}(\mathbb{R})$, the martingale problem for (L_λ, ρ) , with $\mathcal{D}(L_\lambda) = C_c^2(\mathbb{R})$, is well-posed.

Using the theory of Markov functions (see Appendix), the intertwining relation (9) yields the following result of Matsumoto and Yor [17] and Baudoin [1].

Theorem 1 *Let $\rho \in \mathcal{P}(\mathbb{R})$ and $\nu = \rho(dx) \nu_x(du) \in \mathcal{P}(\mathbb{R}^2)$, where*

$$\nu_x(du) = \psi_\lambda(x)^{-1} K_\lambda(x, u) du.$$

Let (X, U) be a diffusion process in \mathbb{R}^2 with initial condition ν and infinitesimal generator A_λ . Then X is a diffusion process in \mathbb{R} with infinitesimal generator L_λ . Moreover, for each $t \geq 0$ and $g \in B(\mathbb{R})$,

$$E[g(U_t) | X_s, 0 \leq s \leq t] = \int_{-\infty}^{\infty} g(u) \nu_{X_t}(du),$$

almost surely.

Proof This follows from the intertwining relation (9), using Theorem 4. The map $\gamma : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $\gamma(x, u) = x$ is continuous and the Markov transition kernel Λ from \mathbb{R} to \mathbb{R}^2 defined by

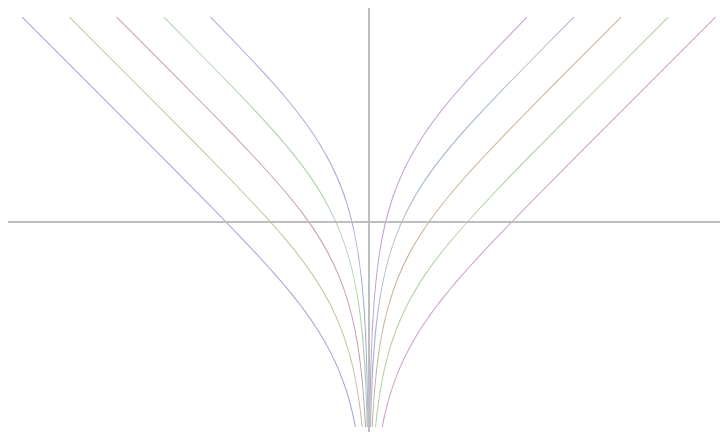
$$\Lambda f(x) = \int_{-\infty}^{\infty} \nu_x(du) f(x, u), \quad f \in B(\mathbb{R})$$

satisfies $\Lambda(g \circ \gamma) = g$ for $g \in B(\mathbb{R})$. Moreover, by (9),

$$L_\lambda \Lambda f = \Lambda A_\lambda f, \quad f \in \mathcal{D}(A_\lambda). \tag{12}$$

Now, $\mathcal{D}(A_\lambda) = C_c^2(\mathbb{R}^2)$ is closed under multiplication, separates points and is convergence determining. Finally, by Itô’s lemma and the intertwining relation (12), the martingale problem for (L_λ, ρ) , now taking $\mathcal{D}(L_\lambda) = \Lambda(\mathcal{D}(A_\lambda)) \cup C_c^2(\mathbb{R})$, is also well-posed, so we are done. \square

Fig. 1 The Toda flow in u (horizontal) and x (vertical) coordinates



To summarise, for any given value of the constant of motion $\lambda = \dot{u} \in \mathbb{R}$, the classical flow in \mathbb{R}^2 is along the curve $\sinh u = \lambda e^x$ (see Fig. 1), according to the evolution equations

$$\dot{u} = \lambda, \quad \dot{x} = \dot{u} + e^{-u-x}, \tag{13}$$

and the x -coordinate satisfies the equation of motion $\ddot{x} = -e^{-2x}$. If we add noise to the constant of motion λ , then the evolution is described by the stochastic differential equations

$$dU = dB + \lambda dt, \quad dX = dU + e^{-U-X} dt \tag{14}$$

and, for appropriate (random) initial conditions, the u -coordinate evolves as a Brownian motion with drift λ and the x -coordinate evolves as a diffusion process in \mathbb{R} with infinitesimal generator L_λ . As (13) is essentially a rewriting of the Bäcklund transformation (3), and in view of Theorem 1, it seems natural to refer to (14) as a *stochastic Bäcklund transformation*, hence the title of this paper.

To relate this to the semi-classical limit, consider the Hamiltonian

$$H^{(\varepsilon)} = \frac{\varepsilon}{2} \partial_x^2 - \frac{1}{\varepsilon} e^{-2x}.$$

Now the eigenfunctions are given by

$$\psi_\lambda^{(\varepsilon)}(x) = \int_{-\infty}^{\infty} K_\lambda(x, u)^{1/\varepsilon} du,$$

and Theorem 1 can be restated as follows. Let $X_0 = x$ and choose U_0 at random according to the probability distribution

$$\nu_x^{(\varepsilon)} = \psi_\lambda^{(\varepsilon)}(x)^{-1} K_\lambda(x, u)^{1/\varepsilon} du.$$

Let (X, U) be the unique solution to the SDE

$$dU = \sqrt{\varepsilon} dB + \lambda dt, \quad dX = dU + e^{-U-X} dt,$$

with this initial condition. Then X is a diffusion process in \mathbb{R} with infinitesimal generator given by

$$\frac{\varepsilon}{2} \partial_x^2 + \varepsilon \partial_x \ln \psi_\lambda^{(\varepsilon)}(x) \cdot \partial_x.$$

As $\varepsilon \rightarrow 0$, the evolution of (X, U) reduces to the evolution equations (7) and the initial distribution of U_0 concentrates on the unique solution $u_\lambda(x)$ to the critical point equation $\partial_u \ln K_\lambda = 0$. On the other hand, one might expect

$$\varepsilon \partial_x \ln \psi_\lambda^{(\varepsilon)}(x) \rightarrow \partial_x [\ln K_\lambda(x, u_\lambda(x))],$$

as is indeed the case, and the evolution of X reduces to the gradient flow

$$\dot{x} = \partial_x [\ln K_\lambda(x, u_\lambda(x))],$$

which is equivalent to (7) thanks to the remarkable identity

$$\partial_x [\ln K_\lambda(x, u_\lambda(x))] = [\partial_x \ln K_\lambda](x, u_\lambda(x)).$$

3 Rational Calogero-Moser System

In this section, we formulate an analogous construction for the one-dimensional rational Calogero-Moser system. Consider the kernel function

$$K(x, u) = \frac{x^2 - u^2}{x}, \quad |u| \leq x,$$

and note that K satisfies

$$(\partial_x \ln K)^2 - (\partial_u \ln K)^2 = 1/x^2 \quad (15)$$

and

$$\partial_x^2 \ln K - \partial_u^2 \ln K = 1/x^2. \quad (16)$$

The corresponding Bäcklund transformation

$$\dot{u} = \frac{1}{x-u} - \frac{1}{x+u} = -\partial_u \ln K, \quad \dot{x} = \frac{1}{x-u} + \frac{1}{x+u} - \frac{1}{x} = \partial_x \ln K \quad (17)$$

has the property that, if (17) holds, then x satisfies the equations of motion of the rational Calogero-Moser system with Hamiltonian

$$\frac{1}{2}p^2 - \frac{1}{2x^2},$$

and $\dot{u} = \lambda$ is a conserved quantity for the coupled system. Indeed, as in the Toda case, differentiating (15) with respect to x and u yields, respectively,

$$\ddot{x} = \partial_x^2 \ln K \partial_x \ln K - \partial_u \partial_x \ln K \partial_u \ln K = -1/x^3, \quad (18)$$

and

$$\ddot{u} = \partial_u^2 \ln K \partial_u \ln K - \partial_x \partial_u \ln K \partial_x \ln K = 0. \quad (19)$$

It also follows from (15) and (17) that λ is an eigenvalue of the Lax matrix

$$\begin{pmatrix} p & 1/x \\ -1/x & -p \end{pmatrix}.$$

As before, $\dot{u} = \lambda$ is equivalent to the critical point equation $\partial_u \ln K_\lambda = 0$, where $K_\lambda = e^{\lambda u} K$. Using this equation, namely

$$2u = \lambda(x^2 - u^2), \quad (20)$$

we can rewrite the evolution equations as

$$\dot{u} = \lambda, \quad \dot{x} = \lambda + \frac{2}{x+u} - \frac{1}{x} = (\partial_x + \partial_u) \ln K_\lambda. \quad (21)$$

The critical point equation (20) has a unique solution $u_\lambda(x) \in (-x, x)$ for any $\lambda \in \mathbb{R}$ and $x > 0$. The relation (20) is stable under the new evolution equation (21), and

is now required to be in force in order to guarantee that (x, p) evolves according to the rational Calogero-Moser flow on the iso-spectral manifold corresponding to λ . Given any $\lambda \in \mathbb{R}$, the evolution equations (21) are well-posed on the corresponding iso-spectral manifold (defined in these coordinates by the relation (20)) in the sense that they admit a unique semi-global solution. For any $\lambda \in \mathbb{R}$ and initial condition $x(0) = x_0 > 0$, the solution is given explicitly for all $t \geq 0$ by

$$u(t) = u_\lambda(x_0) + \lambda t, \quad x(t) = \begin{cases} \sqrt{u(t)^2 + 2u(t)/\lambda} & \lambda \neq 0 \\ \sqrt{x_0^2 + 2t} & \lambda = 0. \end{cases}$$

As in the Toda case, the evolution equation (21) provide the correct framework into which we can introduce noise with the desired outcome.

Let

$$H = \frac{1}{2} \partial_x^2 - \frac{1}{x^2},$$

and write $H_\lambda = H - \lambda^2/2$. Combining (15) and (16) gives the intertwining relation

$$H_\lambda K_\lambda = \left(\frac{1}{2} \partial_u^2 - \lambda \partial_u \right) K_\lambda. \quad (22)$$

It follows that

$$\psi_\lambda(x) = \int_{-x}^x K_\lambda(x, u) du$$

is an eigenfunction of H with eigenvalue $\lambda^2/2$. To see this, first note that

$$\partial_x K_\lambda = e^{\lambda u} \left(1 + \frac{u^2}{x^2} \right), \quad \partial_u K_\lambda = -\frac{2u}{x} e^{\lambda u} + \lambda K_\lambda,$$

and

$$K_\lambda(x, x) = K_\lambda(x, -x) = 0.$$

By the Leibnitz rule,

$$\partial_x \psi_\lambda = \int_{-x}^x \partial_x K_\lambda du + K_\lambda(x, x) + K_\lambda(x, -x) = \int_{-x}^x \partial_x K_\lambda du,$$

and so

$$\begin{aligned} \partial_x^2 \psi_\lambda &= \int_{-x}^x \partial_x^2 K_\lambda du + \partial_x K_\lambda(x, x) + \partial_x K_\lambda(x, -x) \\ &= \int_{-x}^x \partial_x^2 K_\lambda du + 2(e^{\lambda x} + e^{-\lambda x}). \end{aligned}$$

It follows, using (22), that

$$\begin{aligned} H_\lambda \psi_\lambda &= \int_{-x}^x H_\lambda K_\lambda du + (e^{\lambda x} + e^{-\lambda x}) \\ &= \int_{-x}^x \left(\frac{1}{2}\partial_u^2 - \lambda\partial_u\right)K_\lambda du + (e^{\lambda x} + e^{-\lambda x}) \\ &= \left(\frac{1}{2}\partial_u - \lambda\right)K_\lambda \Big|_{u=-x}^{u=x} + (e^{\lambda x} + e^{-\lambda x}) = 0, \end{aligned}$$

as required.

Remark 2 The above integral representation is a special case of the *Dixon-Anderson formula* [7]. The corresponding Bäcklund transformation is a special case of the one introduced in [3], see also [2, 29].

We note that $\psi_0(x) = 2x^2/3$, $\psi_{-\lambda}(x) = \psi_\lambda(x)$ and, for $\lambda > 0$,

$$\psi_\lambda(x) = \lambda^{-3/2} \sqrt{2\pi x} I_{3/2}(\lambda x),$$

where $I_\nu(z)$ is the modified Bessel function of the first kind.

Let

$$D = \{(x, u) \in \mathbb{R}^2 : |u| < x\}.$$

Consider the integral operator defined, for suitable $f : D \rightarrow \mathbb{R}$, by

$$\tilde{K}_\lambda f(x) = \int_{-x}^x K_\lambda(x, u) f(x, u) du,$$

and the differential operator, defined on $\mathcal{D}(A_\lambda) = C_c^2(D)$, by

$$A_\lambda = \frac{1}{2}\partial_x^2 + \frac{1}{2}\partial_u^2 + \partial_x\partial_u + \lambda\partial_u + \left(\lambda + \frac{2}{x+u} - \frac{1}{x}\right)\partial_x.$$

Proposition 2 For $f \in \mathcal{D}(A_\lambda)$,

$$H_\lambda \tilde{K}_\lambda f = \tilde{K}_\lambda A_\lambda f. \tag{23}$$

Proof This follows from (22), as in the proof of Proposition 1. □

Now suppose $\lambda \in \mathbb{R}$. Let B be a standard one-dimensional Brownian motion and consider the coupled stochastic differential equations obtained by adding white noise to λ in (21), that is

$$dU = dB + \lambda dt, \quad dX = dU + \left(\frac{2}{X+U} - \frac{1}{X}\right) dt. \tag{24}$$

Lemma 1 For any initial condition $v \in \mathcal{P}(D)$, the stochastic differential equation (24) has a unique strong solution with continuous sample paths in D . It is a diffusion process in D with infinitesimal generator A_λ and the martingale problem for (A_λ, v) is well-posed.

Proof The function

$$(x, u) \mapsto \left(\lambda + \frac{2}{x+u} - \frac{1}{x}, \lambda \right)$$

is uniformly Lipschitz and bounded on

$$D_\epsilon = \{(x, u) \in D : x + u > \epsilon, x - u > \epsilon\}$$

for any $\epsilon > 0$, so by standard arguments, for any fixed initial condition $(x, u) \in D$, the SDE (24) has a unique strong solution with continuous sample paths up until the first exit time τ from the domain D . We are therefore required to show that $\tau = +\infty$ almost surely. As $X_t - U_t$ is non-decreasing, this is equivalent to showing that $Y_t = X_t + U_t$ almost surely never vanishes. We show this by a simple comparison argument. Set

$$b(x, u) = \frac{2}{x+u} - \frac{1}{x} = \frac{x-u}{x+u} \frac{1}{x},$$

and note that for $(x, u) \in D$ with $x - u \geq \delta$, where $\delta > 0$,

$$b(x, u) > \frac{2}{x+u} - \frac{2}{\delta}.$$

Indeed, if $x \leq \delta/2$ then

$$b(x, u) = \frac{x-u}{x+u} \frac{1}{x} \geq \frac{\delta}{x+u} \frac{2}{\delta} > \frac{2}{x+u} - \frac{2}{\delta};$$

on the other hand, if $x > \delta/2$, then

$$b(x, u) = \frac{2}{x+u} - \frac{1}{x} > \frac{2}{x+u} - \frac{2}{\delta}.$$

Now,

$$dY = 2dU + b(X, U)dt,$$

and it is straightforward to see that the one-dimensional SDE

$$dR = 2dU + \left(\frac{2}{R} - \frac{2}{\delta} \right) dt$$

has a unique strong solution with continuous sample paths in $(0, \infty)$ for any $R_0 = r > 0$; by the usual boundary classification 0 is an entrance boundary for

this diffusion. Thus, if $(X_0, U_0) = (x, u)$ and we set $\delta = x - u$ and $r = x + u$, then $Y_t \geq R_t > 0$ almost surely for all $t \geq 0$, proving the first claim. The second claim follows. \square

Combining this with the intertwining relation (23), we obtain:

Theorem 2 *Let $\rho \in \mathcal{P}((0, \infty))$ and $\nu = \rho(dx)\nu_x(du) \in \mathcal{P}(D)$, where*

$$\nu_x(du) = \psi_\lambda(x)^{-1}K_\lambda(x, u)du.$$

Let (X, U) be a diffusion process in D with initial condition ν and infinitesimal generator A_λ . Then X is a diffusion process in $(0, \infty)$ with infinitesimal generator

$$L_\lambda = \psi_\lambda(x)^{-1}H_\lambda\psi_\lambda(x) = \frac{1}{2}\partial_x^2 + \partial_x \ln \psi_\lambda(x) \cdot \partial_x.$$

Moreover, for each $t \geq 0$ and $g \in B(\mathbb{R})$,

$$E[g(U_t) | X_s, 0 \leq s \leq t] = \int_{-X_t}^{X_t} g(u)\nu_{X_t}(du),$$

almost surely.

Proof This follows from the intertwining relation (23) using Theorem 4. First note that we can identify D with \mathbb{R}^2 via the one-to-one mapping $(x, u) \mapsto (\ln(x + u), \ln(x - u))$ and thus regard D , equipped with the metric induced from the Euclidean metric on \mathbb{R}^2 , as a complete, separable, locally compact metric space. Similarly, we identify $(0, \infty)$ with \mathbb{R} via the one-to-one mapping $x \mapsto \ln x$ and regard $(0, \infty)$, equipped with the metric induced from the Euclidean metric on \mathbb{R} , as a complete, separable metric space. Note that this does not alter the topologies on D and $(0, \infty)$, or the definitions of $B(D)$, $C_b(D)$, $\mathcal{P}(D)$, $C_c^2(D)$, $B((0, \infty))$, $C_b((0, \infty))$, $\mathcal{P}((0, \infty))$, $C_c^2((0, \infty))$, and so on: it is just a smooth change of variables.

The map $\gamma : D \rightarrow (0, \infty)$ defined by $\gamma(x, u) = x$ is continuous and the Markov transition kernel Λ from $(0, \infty)$ to D defined by

$$\Lambda f(x) = \int_{-x}^x \nu_x(du)f(x, u), \quad f \in B(D)$$

satisfies $\Lambda(g \circ \gamma) = g$ for $g \in B((0, \infty))$. Moreover, by (23),

$$L_\lambda \Lambda f = \Lambda A_\lambda f, \quad f \in \mathcal{D}(A_\lambda). \tag{25}$$

Now, $\mathcal{D}(A_\lambda) = C_c^2(D)$ is closed under multiplication, separates points and is convergence determining. Thus, all that remains to be shown is that the martingale problem for (L_λ, ρ) , for some $\mathcal{D}(L_\lambda) \supset \Lambda(\mathcal{D}(A_\lambda))$, is well-posed.

As $\psi_\lambda(x) = \psi_{-\lambda}(x)$, we can assume $\lambda \geq 0$. The drift $b_\lambda(x) = \partial_x \ln \psi_\lambda(x)$ is given by $2/x$ if $\lambda = 0$ and, for $\lambda > 0$,

$$b_\lambda(x) = \frac{1}{2x} + \lambda \frac{I'_{3/2}(\lambda x)}{I_{3/2}(\lambda x)} = \frac{1}{2x} + \lambda \frac{I_{1/2}(\lambda x) + I_{5/2}(\lambda x)}{2I_{3/2}(\lambda x)}.$$

This is bounded below by $1/2x$ and converges to λ as $x \rightarrow +\infty$. In fact, $H_\lambda \psi_\lambda = 0$ implies

$$\partial_x^2 \ln \psi(x) = 2/x^2 - \lambda^2 - b_\lambda(x)^2,$$

hence $b_\lambda(x)$ is uniformly Lipschitz and bounded on (a, ∞) for any $a > 0$. It follows that 0 is an entrance boundary and $+\infty$ is a natural boundary for this one-dimensional diffusion process and the martingale problem for (L_λ, ρ) with $\mathcal{D}(L_\lambda) = C_c^2((0, \infty))$ is well-posed. By Itô's lemma and the intertwining relation (25), we conclude that the martingale problem for (L_λ, ρ) with $\mathcal{D}(L_\lambda) = \Lambda(\mathcal{D}(A_\lambda)) \cup C_c^2((0, \infty))$ is also well-posed, as required. \square

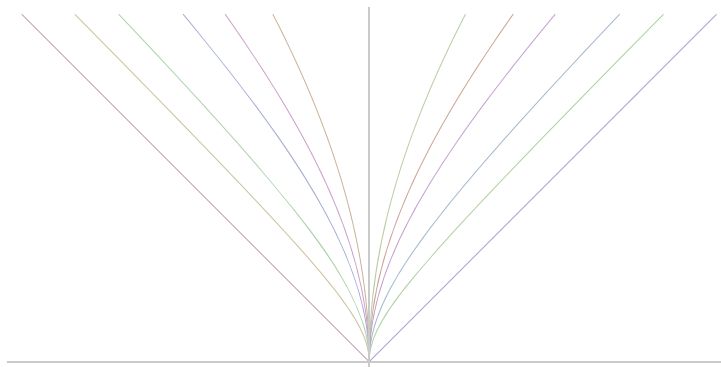


Fig. 2 The rational Calogero-Moser flow in D , shown here with u as the horizontal and x as the vertical coordinate

To summarise, for any given value of the constant of motion $\lambda = \dot{u} \in \mathbb{R}$, the classical flow in D is along the curve $2u = \lambda(x^2 - u^2)$ (see Fig. 2), according to the evolution equations

$$\dot{u} = \lambda, \quad \dot{x} = \dot{u} + \frac{2}{x+u} - \frac{1}{x},$$

and the x -coordinate satisfies the equation of motion $\ddot{x} = -1/x^3$. Adding noise to the constant of motion λ gives the *stochastic Bäcklund transformation*

$$dU = dB + \lambda dt, \quad dX = dU + \left(\frac{2}{X+U} - \frac{1}{X} \right) dt;$$

according to Theorem 2, for appropriate (random) initial conditions, U evolves as a Brownian motion with drift λ and the X evolves as a diffusion process in $(0, \infty)$ with infinitesimal generator L_λ .

When $\lambda = 0$, as $u_0(x) = 0$, the Bäcklund transformation reduces to $\dot{x} = 1/x$, as in the example discussed in the introduction. Note however that in this setting

$$L_0 = \frac{1}{2}\partial_x^2 + \frac{2}{x}\partial_x,$$

and the stochastic differential equations (24) do *not* reduce to the one discussed in the introduction which, for example, gives the simpler construction of the diffusion process with generator L_0 as the solution to the stochastic differential equation

$$dX = dB + \frac{2}{X}dt.$$

To see how the above construction relates to the semi-classical limit, let us introduce a parameter $\mu \geq 1$ and consider

$$H = \frac{1}{2\mu}\partial_x^2 - \frac{1 + \mu}{2x^2}.$$

Then all of the above carries over with K_λ replaced by $(K_\lambda)^\mu$ and

$$\psi_\lambda^{(\mu)}(x) = \int_{-x}^x K_\lambda(x, u)^\mu du.$$

In this setting, Theorem 2 can be restated as follows. Let B be a Brownian motion and (X, U) the unique strong solution in D to

$$dU = \mu^{-1/2}dB + \lambda dt, \quad dX = dU + \left(\frac{2}{X + U} - \frac{1}{X} \right) dt \tag{26}$$

with $X_0 = x > 0$ and U_0 chosen at random in $(-x, x)$ according to

$$\nu_x^{(\mu)}(du) = \psi_\lambda^{(\mu)}(x)^{-1} K_\lambda(x, u)^\mu du.$$

Then X evolves as a diffusion process in $(0, \infty)$ with infinitesimal generator

$$\frac{1}{2\mu}\partial_x^2 + \frac{1}{\mu}\partial_x \ln \psi_\lambda^{(\mu)}(x)\partial_x.$$

When $\mu \rightarrow \infty$, the SDE (26) reduces to the deterministic evolution (21) and the initial distribution $\delta_x \times \nu_x^{(\mu)}$ concentrates on $\delta_x \times \delta_{u_\lambda(x)}$ where $u_\lambda(x)$ is the unique solution in $(-x, x)$ to the critical point equation $\partial_u \ln K_\lambda = 0$ or, equivalently

$2u = \lambda(x^2 - u^2)$. On the other hand, one might expect

$$\frac{1}{\mu} \partial_x \ln \psi_\lambda^{(\mu)}(x) \rightarrow \partial_x [\ln K_\lambda(x, u_\lambda(x))],$$

(which is indeed the case) and so in the limit as $\mu \rightarrow \infty$, the evolution of X is according to the gradient flow

$$\dot{x} = \partial_x [\ln K_\lambda(x, u_\lambda(x))].$$

Comparing this with (21) gives, as in the Toda case,

$$\partial_x [\ln K_\lambda(x, u_\lambda(x))] = [\partial_x \ln K_\lambda](x, u_\lambda(x)),$$

which can be verified directly.

If $-1/2 \leq \mu \leq 1$ and we consider

$$H = \frac{1}{2} \partial_x^2 - \frac{\mu(\mu + 1)}{2x^2},$$

then things are more complicated, because now the evolution

$$dU = dB + \lambda dt, \quad dX = dU + \mu \left(\frac{2}{X + U} - \frac{1}{X} \right) dt$$

can reach the boundary of D and one needs to introduce reflecting boundary conditions on the boundary $x + u = 0$ in the x direction to ensure that the appropriate intertwining relation holds; even then, proving the analogue of Theorem 2 is considerably more technical. One can also consider the case $-3/2 \leq \mu < -1/2$, but then the diffusion with infinitesimal generator L_λ will also require either reflecting (for $\mu > -3/2$) or absorbing (for $\mu = -3/2$) boundary conditions at zero.

Formally it can be seen that the analogue of Theorem 2, in the case $\mu = 0$, corresponds to Pitman's '2M-X' theorem, for general drift and initial condition [25, 26], which can be stated as follows. Let $x \geq 0$ and U be a Brownian motion with drift λ and U_0 chosen at random in $[-x, x]$ with probability density proportional to $e^{\lambda u}$. Set

$$X_t = U_t - \min\{2 \inf_{s \leq t} U_s, U_0 - x\}, \quad t \geq 0.$$

Then (X, U) is a reflected Brownian motion (with singular covariance) in the closure of D and X is a diffusion process in $[0, \infty)$ started at x with infinitesimal generator

$$\frac{1}{2} \partial_x^2 + \lambda \coth(\lambda x) \partial_x.$$

4 Hyperbolic Calogero-Moser System I

The above example extends to the hyperbolic case, taking

$$K(x, u) = \left[\frac{\sinh\left(\epsilon \frac{x+u}{2}\right) \sinh\left(\epsilon \frac{x-u}{2}\right)}{\sinh \epsilon x} \right]^\mu, \quad |u| < x.$$

We will assume for convenience that $\mu \geq 1$. Now,

$$(\partial_x \ln K)^2 - (\partial_u \ln K)^2 = \frac{\epsilon^2 \mu^2}{\sinh^2 \epsilon x} \quad (27)$$

and

$$\partial_x^2 \ln K - \partial_u^2 \ln K = \frac{\epsilon^2 \mu}{\sinh^2 \epsilon x}. \quad (28)$$

The corresponding Bäcklund transformation

$$\dot{u} = -\partial_u \ln K, \quad \dot{x} = \partial_x \ln K \quad (29)$$

agrees with the one given in [29] and has the property that, if (29) holds, then x satisfies the equations of motion of the hyperbolic Calogero-Moser system with Hamiltonian

$$\frac{1}{2}p^2 - \frac{\epsilon^2 \mu^2}{2 \sinh^2 \epsilon x}, \quad (30)$$

and $\dot{u} = \lambda$ is a conserved quantity for the coupled system. Indeed, as before, differentiating (27) with respect to x and u yields, respectively,

$$\ddot{x} = \partial_x^2 \ln K \partial_x \ln K - \partial_u \partial_x \ln K \partial_u \ln K = \partial_x \frac{\epsilon^2 \mu^2}{2 \sinh^2 \epsilon x}, \quad (31)$$

and

$$\ddot{u} = \partial_u^2 \ln K \partial_u \ln K - \partial_x \partial_u \ln K \partial_x \ln K = 0. \quad (32)$$

It also follows from (27) and (29) that λ is an eigenvalue of the Lax matrix

$$\begin{pmatrix} p & \epsilon \mu / \sinh \epsilon x \\ -\epsilon \mu / \sinh \epsilon x & -p \end{pmatrix}. \quad (33)$$

The equation $\dot{u} = \lambda$ is equivalent to the critical point equation $\partial_u \ln K_\lambda = 0$. Using this equation, namely

$$\coth\left(\epsilon \frac{x-u}{2}\right) - \coth\left(\epsilon \frac{x+u}{2}\right) = \frac{2\lambda}{\epsilon\mu}, \quad (34)$$

we can rewrite the evolution equation (29) as

$$\dot{u} = \lambda, \quad \dot{x} = \lambda + b(x, u) = (\partial_x + \partial_u) \ln K_\lambda, \quad (35)$$

where

$$b(x, u) = (\partial_x + \partial_u) \ln K = \mu\epsilon \left[\coth\left(\epsilon \frac{x+u}{2}\right) - \coth \epsilon x \right].$$

Equation (34) has a unique solution $u_\lambda(x) \in \mathbb{R}$ for each $x > 0$ and $\lambda \in \mathbb{R}$. The relation (34) is stable under the new evolution equations (35), and is now required to be in force in order to guarantee that (x, p) evolves according to the hyperbolic Calogero-Moser flow on the iso-spectral manifold corresponding to λ .

Note that $u_0(x) = 0$ for all $x > 0$, so when $\lambda = 0$, we must have $u(t) = 0$ for all $t \geq 0$ and the equation for x simplifies to $\dot{x} = \epsilon\mu / \sinh \epsilon x$, which admits a unique semi-global solution for any initial condition $x(0) = x_0 > 0$, defined for all $t \geq 0$ by

$$x(t) = \frac{1}{\epsilon} \cosh^{-1}(\cosh \epsilon x_0 + \epsilon^2 \mu t). \quad (36)$$

For $\lambda \in \mathbb{R} \setminus \{0\}$, the function u_λ is a bijection from $(0, \infty)$ to \mathbb{R} , with inverse given by

$$u_\lambda^{-1}(u) = \frac{2}{\epsilon} \cosh^{-1} \sqrt{\frac{\epsilon\mu}{2\lambda} \sinh \epsilon u + \cosh^2 \frac{\epsilon u}{2}}.$$

It follows that, for any given $\lambda \in \mathbb{R}$, the evolution equations (35) are well-posed on the corresponding iso-spectral manifold (defined in these coordinates by the relation (34)) in the sense that they admit a unique semi-global solution. For $\lambda \in \mathbb{R} \setminus \{0\}$ and initial condition $x(0) = x_0 > 0$, the solution is given explicitly for all $t \geq 0$ by $u(t) = u_\lambda(x_0) + \lambda t$ and $x(t) = u_\lambda^{-1}(u(t))$. As before, the equations (35) provide the correct framework into which we can introduce noise with the desired outcome.

Combining (27) and (28) gives the intertwining relation

$$H_\lambda K_\lambda = \left(\frac{1}{2} \partial_u^2 - \lambda \partial_u\right) K_\lambda \quad (37)$$

where $K_\lambda = e^{\lambda u}K$, $H_\lambda = H - \lambda^2/2$, and

$$H = \frac{1}{2}\partial_x^2 - \frac{\epsilon^2\mu(\mu + 1)}{2\sinh^2 \epsilon x}. \tag{38}$$

As before, it follows, using (37) and the Leibnitz rule, that

$$\psi_\lambda(x) = \int_{-x}^x K_\lambda(x, u)du$$

is an eigenfunction of H with eigenvalue $\lambda^2/2$. Indeed, if $\mu > 1$, then K_λ , $\partial_x K_\lambda$ and $\partial_u K_\lambda$ vanish for $u = \pm x$ and the claim is immediate. If $\mu = 1$, then

$$\begin{aligned} \partial_x K_\lambda(x, x) &= \frac{\epsilon}{2}e^{\lambda x}, & \partial_x K_\lambda(x, -x) &= \frac{\epsilon}{2}e^{-\lambda x}, \\ \partial_u K_\lambda(x, x) &= -\frac{\epsilon}{2}e^{\lambda x}, & \partial_u K_\lambda(x, -x) &= \frac{\epsilon}{2}e^{-\lambda x} \end{aligned}$$

and

$$K_\lambda(x, x) = K_\lambda(x, -x) = 0.$$

By the Leibnitz rule,

$$\partial_x \psi_\lambda = \int_{-x}^x \partial_x K_\lambda du + K_\lambda(x, x) + K_\lambda(x, -x) = \int_{-x}^x \partial_x K_\lambda du,$$

and so

$$\begin{aligned} \partial_x^2 \psi_\lambda &= \int_{-x}^x \partial_x^2 K_\lambda du + \partial_x K_\lambda(x, x) + \partial_x K_\lambda(x, -x) \\ &= \int_{-x}^x \partial_x^2 K_\lambda du + \frac{\epsilon}{2}(e^{\lambda x} + e^{-\lambda x}). \end{aligned}$$

It follows, using (22), that

$$\begin{aligned} H_\lambda \psi_\lambda &= \int_{-x}^x H_\lambda K_\lambda du + \frac{\epsilon}{4}(e^{\lambda x} + e^{-\lambda x}) \\ &= \int_{-x}^x \left(\frac{1}{2}\partial_u^2 - \lambda\partial_u\right)K_\lambda du + \frac{\epsilon}{4}(e^{\lambda x} + e^{-\lambda x}) \\ &= \left(\frac{1}{2}\partial_u - \lambda\right)K_\lambda \Big|_{u=-x}^{u=x} + \frac{\epsilon}{4}(e^{\lambda x} + e^{-\lambda x}) = 0, \end{aligned}$$

as required.

For example, when $\mu = 1$,

$$\psi_\lambda(x) = \frac{\epsilon}{\epsilon^2 - \lambda^2} \left[\frac{\epsilon}{\lambda} \coth \epsilon x \sinh \lambda x - \cosh \lambda x \right].$$

In particular, $\psi_0(x) = x \coth \epsilon x - 1/\epsilon$.

Continuing as before, this kernel function leads to a hyperbolic version of Theorem 2, valid for any $\lambda \in \mathbb{R}$.

5 Hyperbolic Calogero-Moser System II

There is another choice of kernel function which leads to a very different 'version' of Theorem 2, valid only for a restricted range of λ . It is based on the kernel functions considered in [9, 10] and, in the rational case, reduces to the example discussed in the introduction.

Let $D = (0, \infty) \times \mathbb{R}$ and consider the kernel function

$$K(x, u) = \left[\tanh \left(\epsilon \frac{x+u}{2} \right) + \tanh \left(\epsilon \frac{x-u}{2} \right) \right]^\mu, \quad (x, u) \in D.$$

Note that we can also write

$$K(x, u) = \left[\frac{\sinh \epsilon x}{\cosh(\epsilon(x+u)/2) \cosh(\epsilon(x-u)/2)} \right]^\mu.$$

Now,

$$(\partial_x \ln K)^2 - (\partial_u \ln K)^2 = \frac{\epsilon^2 \mu^2}{\sinh^2 \epsilon x} \quad (39)$$

and

$$\partial_x^2 \ln K - \partial_u^2 \ln K = -\frac{\epsilon^2 \mu}{\sinh^2 \epsilon x}. \quad (40)$$

The corresponding Bäcklund transformation

$$\dot{u} = -\partial_u \ln K, \quad \dot{x} = \partial_x \ln K \quad (41)$$

has the property that, if (41) holds, then x satisfies the equations of motion of the hyperbolic Calogero-Moser system with Hamiltonian (30) and $\dot{u} = \lambda$ is a conserved quantity for the coupled system, as can be seen by differentiating (39) with respect to x and u , respectively. It also follows from (39) and (41) that λ is an eigenvalue of the Lax matrix (33).

Now the equation $\dot{u} = \lambda$ is equivalent to the critical point equation $\partial_u \ln K_\lambda = 0$, where $K_\lambda = e^{\lambda u} K$. Using this equation, namely

$$\tanh\left(\epsilon \frac{x+u}{2}\right) - \tanh\left(\epsilon \frac{x-u}{2}\right) = \frac{2\lambda}{\epsilon\mu}, \tag{42}$$

we can rewrite the evolution equation (29) as

$$\dot{u} = \lambda, \quad \dot{x} = \lambda + b(x, u) = (\partial_x + \partial_u) \ln K_\lambda, \tag{43}$$

where

$$b(x, u) = (\partial_x + \partial_u) \ln K = \mu\epsilon \left[\coth \epsilon x - \tanh\left(\epsilon \frac{x+u}{2}\right) \right].$$

In this setting, the critical point equation (42) only has a solution $u_\lambda(x) \in \mathbb{R}$ if $|\lambda| < \mu\epsilon$, in which case it is unique. We note that $u_0(x) = 0$ for all $x > 0$ and $u_\lambda(x) \rightarrow \pm\infty$ when $\lambda \rightarrow \pm\mu\epsilon$. The relation (42) is stable under the new evolution equations (43), and is now required to be in force in order to guarantee that (x, p) evolves according to the hyperbolic Calogero-Moser flow on the iso-spectral manifold corresponding to λ .

When $\lambda = 0$, we must have $u(t) = 0$ for all $t \geq 0$ and the equation for x simplifies to $\dot{x} = \epsilon\mu / \sinh \epsilon x$, as in the previous example, which admits a unique solution for any initial condition $x(0) = x_0 > 0$, defined for all $t \geq 0$ by (36).

For $\lambda > 0$, the function u_λ is a bijection from $(0, \infty)$ to $(0, \infty)$, with inverse

$$u_\lambda^{-1}(u) = \frac{2}{\epsilon} \cosh^{-1} \sqrt{\frac{\epsilon\mu}{2\lambda} \sinh \epsilon u - \sinh^2 \frac{\epsilon u}{2}}.$$

Note that the constraint $\lambda < \epsilon\mu$ ensures that the quantity in the square root is positive. For $\lambda < 0$, u_λ is a bijection from $(0, \infty)$ to $(-\infty, 0)$, with inverse given by the same formula. It follows that, given any $\lambda \in \mathbb{R}$, the evolution equations (43) are well-posed on the corresponding iso-spectral manifold (defined in these coordinates by the relation (42)) in the sense that they admit a unique semi-global solution. For $\lambda \in \mathbb{R} \setminus \{0\}$ and initial condition $x(0) = x_0 > 0$, the solution is given for all $t \geq 0$ by $u(t) = u_\lambda(x_0) + \lambda t$ and $x(t) = u_\lambda^{-1}(u(t))$. As before, the evolution equations (43) provide the correct framework into which we can introduce noise with the desired outcome.

Now let

$$H = \frac{1}{2} \partial_x^2 - \frac{\epsilon^2 \mu (\mu - 1)}{2 \sinh^2 \epsilon x},$$

and write $H_\lambda = H - \lambda^2/2$. Note that this Hamiltonian has a different coupling constant to the one in (38), reflecting the difference between (40) and (28).

Combining (39) and (40) gives the intertwining relation

$$H_\lambda K_\lambda = \left(\frac{1}{2}\partial_u^2 - \lambda\partial_u\right)K_\lambda \quad (44)$$

and it follows, using the Leibnitz rule, that for $|\operatorname{Re} \lambda| < \epsilon\mu$,

$$\psi_\lambda(x) = \int_{-\infty}^{\infty} K_\lambda(x, u)du$$

is an eigenfunction of H with eigenvalue $\lambda^2/2$.

As noted in [10, Eq. (4.16)], the eigenfunction ψ_λ is related to the associated Legendre function of the first kind by

$$\psi_\lambda(x) = \frac{2^{2\mu+3/2}}{\sqrt{\pi}\epsilon} (\sinh \epsilon x)^{1/2} \frac{\Gamma(\mu + \lambda/\epsilon)\Gamma(\mu - \lambda/\epsilon)}{\Gamma(\mu)} P_{\frac{\lambda}{\epsilon}-\frac{1}{2}}^{\frac{1}{2}-\mu}(\cosh \epsilon x). \quad (45)$$

We note also that $\psi_\lambda(x) = \psi_{-\lambda}(x)$, as can be seen, for example, from the functional equation $P_{-b}^a(z) = P_{b-1}^a(z)$, and

$$\psi_0(x) = \frac{2\sqrt{\pi}\Gamma(\mu)}{\epsilon\Gamma(\mu + 1/2)} (\sinh \epsilon x)^\mu.$$

These are *not* the same eigenfunctions which were obtained in the previous section, even taking account of the different coupling constants. For example, taking $\mu = 2$ here gives $\psi_0(x) = 8(\sinh \epsilon x)^2/3\epsilon$, which is different from the eigenfunction $\psi_0(x) = x \coth \epsilon x - 1/\epsilon$ of the previous section with $\mu = 1$; both are positive on $(0, \infty)$, vanish at zero, and satisfy

$$\psi_0'' - \frac{\epsilon^2}{\sinh^2 \epsilon x} \psi_0 = 0,$$

but they are not equal. On the other hand, they agree (up to a constant factor) in the limit as $\epsilon \rightarrow 0$, which corresponds to the rational case.

Consider the integral operator defined, for suitable $f : D \rightarrow \mathbb{R}$, by

$$\tilde{K}_\lambda f(x) = \int_{-\infty}^{\infty} K_\lambda(x, u)f(x, u)du,$$

and the differential operator, defined on $\mathcal{D}(A_\lambda) = C_c^2(D)$, by

$$A_\lambda = \frac{1}{2}\partial_x^2 + \frac{1}{2}\partial_u^2 + \partial_x\partial_u + \lambda\partial_u + (\lambda + b(x, u))\partial_x.$$

Proposition 3 For $|\operatorname{Re} \lambda| < \epsilon\mu$ and $f \in \mathcal{D}(A_\lambda)$,

$$H_\lambda \tilde{K}_\lambda f = \tilde{K}_\lambda A_\lambda f. \tag{46}$$

Proof This follows from (44), as in the proof of Proposition 1. □

Now, let B be a standard one-dimensional Brownian motion and consider the coupled stochastic differential equations obtained by adding white noise to λ in (7), that is

$$dU = dB + \lambda dt, \quad dX = dU + b(X, U)dt. \tag{47}$$

Lemma 2 Suppose $\lambda \in \mathbb{R}$ with $|\lambda| < \epsilon\mu$ and $\mu \geq 1/2$. For any initial condition $v \in \mathcal{P}(D)$, the stochastic differential equation (47) has a unique strong solution with continuous sample paths in D . It is a diffusion process in D with infinitesimal generator A_λ and the martingale problem for (A_λ, v) is well-posed.

Proof The function $(x, u) \mapsto (\lambda + b(x, u), \lambda)$ is uniformly Lipschitz and bounded on $D_\delta = \{(x, u) \in D : x > \delta\}$ for any $\delta > 0$, so by standard arguments, for any fixed initial condition $(x, u) \in D$, the SDE (47) has a unique strong solution with continuous sample paths up until the first exit time τ from the domain D . We are therefore required to show that $\tau = +\infty$ almost surely or equivalently, that X_t almost surely never vanishes. We show this by a comparison argument, using the fact that on D we have

$$b(x, u) > \mu\epsilon(\coth \epsilon x - 1).$$

Now,

$$dX = dU + b(X, U)dt,$$

and it is straightforward to see that the one-dimensional SDE

$$dR = dU + \mu\epsilon(\coth(\epsilon R) - 1)dt$$

has a unique strong solution with continuous sample paths in $(0, \infty)$ for any $R_0 = r > 0$; since $\mu \geq 1/2$, by the usual boundary classification 0 is an entrance boundary for this diffusion. Thus, if $(X_0, U_0) = (x, u) \in D$ and $R_0 = x - u$, then $X_t \geq R_t > 0$ almost surely for all $t \geq 0$, as required, proving the first claim. The second claim follows. □

Combining this with the intertwining relation (46), we obtain:

Theorem 3 Suppose $\lambda \in \mathbb{R}$ with $|\lambda| < \epsilon\mu$ and $\mu > 1/2$. Let $\rho \in \mathcal{P}((0, \infty))$ and $v = \rho(dx)v_x(du) \in \mathcal{P}(D)$, where $v_x(du) = \psi_\lambda(x)^{-1}K_\lambda(x, u)du$. Let (X, U) be a diffusion process in D with initial condition v and infinitesimal generator A_λ . Then

X is a diffusion process in $(0, \infty)$ with infinitesimal generator

$$L_\lambda = \psi_\lambda(x)^{-1} H_\lambda \psi_\lambda(x) = \frac{1}{2} \partial_x^2 + \partial_x \ln \psi_\lambda(x) \cdot \partial_x.$$

Moreover, for each $t \geq 0$ and $g \in B(\mathbb{R})$,

$$E[g(U_t) | X_s, 0 \leq s \leq t] = \int_{-\infty}^{\infty} g(u) \nu_{X_t}(du),$$

almost surely.

Proof This follows from the intertwining relation (44) using Theorem 4. As before, we identify D with \mathbb{R}^2 via the one-to-one mapping $(x, u) \mapsto (\ln x, u)$ and thus regard D , equipped with the metric induced from the Euclidean metric on \mathbb{R}^2 , as a complete, separable, locally compact metric space. Similarly, we identify $(0, \infty)$ with \mathbb{R} via the one-to-one mapping $x \mapsto \ln x$ and regard $(0, \infty)$, equipped with the metric induced from the Euclidean metric on \mathbb{R} , as a complete, separable metric space.

The map $\gamma : D \rightarrow (0, \infty)$ defined by $\gamma(x, u) = x$ is continuous and the Markov transition kernel Λ from $(0, \infty)$ to D defined by

$$\Lambda f(x) = \int_{-\infty}^{\infty} \nu_x(du) f(x, u), \quad f \in B(D)$$

satisfies $\Lambda(g \circ \gamma) = g$ for $g \in B((0, \infty))$. Moreover, by (46),

$$L_\lambda \Lambda f = \Lambda A_\lambda f, \quad f \in \mathcal{D}(A_\lambda). \tag{48}$$

Now, $\mathcal{D}(A_\lambda) = C_c^2(D)$ is closed under multiplication, separates points and is convergence determining. Thus, all that remains to be shown is that the martingale problem for (L_λ, ρ) , for some $\mathcal{D}(L_\lambda) \supset \Lambda(\mathcal{D}(A_\lambda))$, is well-posed.

By (45) and the relation

$$(z^2 - 1) \frac{d}{dz} P_b^a(z) = bzP_b^a(z) - (a + b)P_{b-1}^a(z),$$

the drift $b_\lambda(x) = \partial_x \ln \psi_\lambda(x)$ is given by

$$b_\lambda(x) = \lambda \coth \epsilon x + \frac{\epsilon \mu - \lambda}{\sinh \epsilon x} \left[P_{\frac{\lambda}{\epsilon} - \frac{3}{2}}^{\frac{1}{2} - \mu}(\cosh \epsilon x) / P_{\frac{\lambda}{\epsilon} - \frac{1}{2}}^{\frac{1}{2} - \mu}(\cosh \epsilon x) \right].$$

As $x \rightarrow 0^+$,

$$P_{\frac{\lambda}{\epsilon} - \frac{3}{2}}^{\frac{1}{2} - \mu}(\cosh \epsilon x) / P_{\frac{\lambda}{\epsilon} - \frac{1}{2}}^{\frac{1}{2} - \mu}(\cosh \epsilon x) \rightarrow 1.$$

Now $\mu > 1/2$, so this implies that $b_\lambda(x) > 1/2x$ for x sufficiently small, which classifies 0 as an entrance boundary. On the other hand, as $x \rightarrow +\infty$,

the second term vanishes and $b_\lambda(x) \rightarrow \lambda$, which shows that $+\infty$ is a natural boundary. The relevant asymptotics can be found, for example, in [19, §14.8.7, §14.8(iii)]. Thus, as b_λ is locally Lipschitz, the martingale problem for (L_λ, ρ) with $\mathcal{D}(L_\lambda) = C_c^2((0, \infty))$ is well-posed. By Itô's lemma and the intertwining relation (25), it follows that the martingale problem for (L_λ, ρ) with $\mathcal{D}(L_\lambda) = \Lambda(\mathcal{D}(A_\lambda)) \cup C_c^2((0, \infty))$ is also well-posed, as required. \square

To summarise, for any given value of the constant of motion $\lambda = \dot{u} \in \mathbb{R}$ with $|\lambda| \leq \mu\epsilon$, the classical flow in D evolves according to the evolution equations

$$\dot{u} = \lambda, \quad \dot{x} = \dot{u} + b(x, u).$$

If we add noise to the constant of motion λ , then the evolution is described by the stochastic Bäcklund transformation

$$dU = dB + \lambda dt, \quad dX = dU + b(X, U)dt$$

and, for appropriate (random) initial conditions, U evolves as a Brownian motion with drift λ and X evolves as a diffusion process in $(0, \infty)$ with infinitesimal generator L_λ .

As in the previous examples, we can let $\mu \rightarrow \infty$ to study the semi-classical limit and the result is analogous. As before, if $\mu = 1$ and $|\lambda| < \epsilon$, and $u_\lambda(x)$ denotes the unique solution to the critical point equation $\partial_u \ln K_\lambda = 0$, then

$$\partial_x [\ln K_\lambda(x, u_\lambda(x))] = [\partial_x \ln K_\lambda](x, u_\lambda(x)).$$

It is natural to ask what happens to the statement of Theorem 3 when $\lambda \rightarrow \mu\epsilon$. In this limit, $b_\lambda(x) \rightarrow \mu\epsilon \coth \epsilon x$ and

$$\Gamma(\mu - \lambda/\epsilon)^{-1} \psi_\lambda(x) \rightarrow \frac{2^{\mu+2} \Gamma(2\mu)}{\sqrt{\pi} \epsilon \Gamma(\mu) \Gamma(\mu + 1/2)} (\sinh \epsilon x)^\mu =: \tilde{\psi}_{\mu\epsilon}(x).$$

Furthermore, since $u_\lambda(x) \rightarrow +\infty$, it is easy to see that the measure ν_x concentrates at $+\infty$. Now, when $\lambda \rightarrow \mu\epsilon$ and $u \rightarrow +\infty$, $\lambda + b(x, u) \rightarrow \mu\epsilon \coth \epsilon x$. The Bäcklund transformation simplifies: if

$$\dot{x} = \mu\epsilon \coth \epsilon x$$

then x evolves according to the hyperbolic Calogero-Moser flow with the constant to motion $\lambda = \mu\epsilon$. The statement of Theorem 3 carries over trivially: if X evolves according to the SDE

$$dX = dB + \mu\epsilon \coth(\epsilon X)dt$$

then X is a diffusion process on $(0, \infty)$ with infinitesimal generator

$$L_{\mu\epsilon} = \tilde{\psi}_{\mu\epsilon}(x)^{-1} H_{\mu\epsilon} \tilde{\psi}_{\mu\epsilon}(x) = \frac{1}{2} \partial_x^2 + \mu\epsilon \coth \epsilon x \cdot \partial_x.$$

Similar remarks apply when $\lambda \rightarrow -\mu\epsilon$, and in fact the limiting statements are the same. When $\epsilon \rightarrow 0$, this reduces to the example discussed in the introduction, and now we can see from the fundamental restriction $|\lambda| < \epsilon\mu$ that in fact we can only hope for the above structure to remain intact in this limit when $\lambda = 0$, as indeed it does with the evolution of the x -coordinate in (43) becoming autonomous and reducing to $\dot{x} = \mu/x$, and the analogue of Theorem 3 carrying over trivially.

6 The KPZ Equation and Semi-infinite Toda Chain

As remarked in the introduction, most of the above constructions extend naturally to higher rank systems. For the n -particle Toda chain, this has been developed in the papers [21, 22]. The construction given in [21] is related to the geometric RSK correspondence. In [23] it was extended to a semi-infinite setting and related to the Kardar-Parisi-Zhang (or stochastic heat) equation. In this context it can be represented formally as a semi-infinite system of coupled stochastic partial differential equations, the first of which is the stochastic heat equation. In the language of the present paper, the construction given in [23] is a stochastic Bäcklund transformation and should be related (in a way that has yet to be fully understood) to a semi-infinite version of the quantum Toda chain. See also [4, 18] for further related work in this direction.

With this picture in mind, it is natural to expect the construction given in [23], without noise, to be related to the semi-infinite classical Toda chain. This is indeed the case, as we will now explain directly. The conclusion is that the fixed-time solution to the KPZ equation, with ‘narrow wedge’ initial condition, can be viewed as the trajectory of the first particle in a stochastic perturbation of a particular solution to the semi-infinite Toda chain.

The stochastic heat equation can be written formally as

$$u_t = \frac{1}{2} u_{xx} + \xi u$$

where $\xi(t, x)$ is space-time white noise. It is related to the KPZ equation

$$h_t = \frac{1}{2} h_{xx} + \frac{1}{2} (h_x)^2 + \xi$$

via the Cole-Hopf transformation $h = \log u$. The extension given in [23] starts with a solution $u(t, x, y)$ to the stochastic heat equation with delta initial condition $u(0, x, y) = \delta(x - y)$ and defines a sequence of ‘ τ -functions’ τ_n which can be

expressed formally as the bi-Wronskians

$$\tau_n = \det[\partial_x^{i-1} \partial_y^{j-1} u]_{i,j=1,\dots,n}.$$

Their evolution can be described, again formally, by the coupled equations

$$\partial_t a_n = \frac{1}{2} \partial_x^2 a_n + \partial_x [a_n \partial_x h_n]$$

where $a_n = \tau_{n-1} \tau_{n+1} / \tau_n^2$ and $h_n = \log(\tau_n / \tau_{n-1})$ with the convention $\tau_0 = 1$. Moreover, formally it can be seen that the τ_n are τ -functions for the 2d Toda chain, that is, $(\ln \tau_n)_{xy} = a_n$.

If we switch off the noise by setting $\xi = 0$, then u is given by the heat kernel

$$u(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$$

and

$$\tau_n = t^{-n(n-1)/2} \left(\prod_{j=1}^{n-1} j! \right) u^n.$$

Note that $a_n = \tau_{n-1} \tau_{n+1} / \tau_n^2 = n/t$ and

$$h_{n+1} = -(x - y)^2/2t - \ln \left[\sqrt{2\pi t} \frac{t^n}{n!} \right].$$

These τ_n satisfy the 2d Toda equations $(\ln \tau_n)_{xy} = a_n$ as before, but now it also holds that $(\ln \tau_n)_{xx} = -a_n$ or, equivalently,

$$(h_n)_{xx} = e^{h_n - h_{n-1}} - e^{h_{n+1} - h_n}$$

(with $h_0 \equiv +\infty$) which are the equations of motion of the semi-infinite Toda chain.

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Appendix

The theory of Markov functions is concerned with the question: when does a function of a Markov process inherit the Markov property? The simplest case is when there is symmetry in the problem, for example, the norm of Brownian motion

in \mathbb{R}^n has the Markov property, for any initial condition, because the heat kernel in \mathbb{R}^n is invariant under rotations. A more general formulation of this idea is the well-known *Dynkin criterion* [5]. There is another, more subtle, criterion which has been proved at various levels of generality by, for example, Kemeny and Snell [13], Rogers and Pitman [26] and Kurtz [14]. It can be interpreted as a time-reversal of Dynkin’s criterion [12] and provides sufficient conditions for a function of a Markov process to have the Markov property, but only for very particular initial conditions. For our purposes, the martingale problem formulation of Kurtz [14] is best suited, as it is quite flexible and formulated in terms of infinitesimal generators.

Let E be a complete, separable metric space. Let $A : \mathcal{D}(A) \subset B(E) \rightarrow B(E)$ and $\nu \in \mathcal{P}(E)$. A progressively measurable E -valued process $X = (X_t, t \geq 0)$ is a solution to the *martingale problem* for (A, ν) if X_0 is distributed according to ν and there exists a filtration \mathcal{F}_t such that

$$f(X_t) - \int_0^t Af(X_s)ds$$

is a \mathcal{F}_t -martingale, for all $f \in \mathcal{D}(A)$. The martingale problem for (A, ν) is *well-posed* if there exists a solution X which is unique in the sense that any two solutions have the same finite-dimensional distributions.

The following is a special case of Corollary 3.5 (see also Theorems 2.6, 2.9 and the remark at the top of page 5) in the paper [14].

Theorem 4 (Kurtz [14]) *Assume that E is locally compact, that $A : \mathcal{D}(A) \subset C_b(E) \rightarrow C_b(E)$, and that $\mathcal{D}(A)$ is closed under multiplication, separates points and is convergence determining. Let F be another complete, separable metric space, $\gamma : E \rightarrow F$ continuous and $\Lambda(y, dx)$ a Markov transition kernel from F to E such that $\Lambda(g \circ \gamma) = g$ for all $g \in B(F)$, where $\Lambda f(x) = \int_E f(x) \Lambda(y, dx)$ for $f \in B(E)$. Let $B : \mathcal{D}(B) \subset B(F) \rightarrow B(F)$, where $\Lambda(\mathcal{D}(A)) \subset \mathcal{D}(B)$, and suppose*

$$B\Lambda f = \Lambda Af, \quad f \in \mathcal{D}(A).$$

Let $\mu \in \mathcal{P}(F)$ and set $\nu = \int_F \mu(dy) \Lambda(y, dx) \in \mathcal{P}(E)$. Suppose that the martingale problems for (A, ν) and (B, μ) are well-posed, and that X is a solution to the martingale problem for (A, ν) . Then $Y = \gamma \circ X$ is a Markov process and a solution to the martingale problem for (B, μ) . Furthermore, for each $t \geq 0$ and $g \in B(F)$ we have, almost surely,

$$E[g(X_t) | Y_s, 0 \leq s \leq t] = \int_E g(x) \Lambda(Y_t, dx).$$

We remark that, under the hypotheses of the above theorem, X is a Markov process and the forward equation

$$\nu_t f = \nu f + \int_0^t \nu_s Af ds, \quad f \in \mathcal{D}(A)$$

has a unique continuous solution in $\mathcal{P}(D)$; also the assumption of uniqueness for the martingale problem for (B, μ) is not necessary, as it is implied by the other hypotheses; we refer the reader to [14] for more details.

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