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Collision times and exit times from cones: A duality

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We consider the first collision time for a set of independent one-dimensional zero-drift Wiener processes. For the 3-process problem, the first collision time corresponds to the first exit time of Brownian motion in a cone in $\mathbb{R}^2$, and we can apply the results of Spitzer (1958) and Dante DeBlassie (1987) to obtain its distribution. In the case where the processes have equal infinitesimal variance, a more elementary method yields nice closed-form results for the 3-process problem, and second order approximations for the general n-process problem. This case (for three processes) corresponds to Brownian motion in a cone of angle $\frac{1}{3}\pi$. The latter approach can in fact be applied to any system of independent (identical) Markov processes, provided the single-barrier hitting time distributions are known for the individual processes and their differences, and provided the processes can't jump over each other.

first exit times * collision times * particle systems * cones

1. Introduction

In this paper we explore the properties of τ, the first collision time for a set of independent one-dimensional Wiener processes/particles, where the first collision time is defined to be the first time at which any two particles collide, and establish the duality between this problem and that of determining the distribution of the first exit time for Brownian motion in a cone. Such a duality was anticipated by Arratia (1979), in his PhD thesis, and here we make his ideas more explicit. The cone problem has been treated by Spitzer (1958) and DeBlassie (1987).

This connection, once established, provides new results for both problems. The results of Spitzer and DeBlassie carry over to the collsiion problem in its most general form. In the collision problem, the special case when the processes have equal variance is treated by a direct method, which in turn provides an explicit result for the corresponding cone problem (where no *explicit* results were provided by the works of Spitzer and DeBlassie).

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The ideas and methods used in this paper may also be helpful in treating aspects of other collision problems, many of which have been considered in the past. For example, collision probabilities for randomly touching particles and bodies have been investigated in Wiel (1979, 1981, 1989), Papaderou-Vogiatzaki (1983), Papaderou-Vogiatzaki and Schneider (1988), and Enns et al. (1984). For a survey on collision probabilities for convex bodies, see Schneider and Wieacker (1984). These authors have generally treated the problem of randomly touching convex bodies, or random subspaces touching a convex body. Hill and Gulati (1980) calculated collision probabilites for two random walking particles, with special consideration given to the location of collision. One-dimensional systems of point-particles with collisions have been investigated by Duerr et al. (1987) and Sznitman (1989); systems of non-colliding point-particles by Karlin (1968); and systems of coalescing and annihilating random walks by Arratia (1981) and Cox (1989).

The structure of the paper is as follows. We will first consider the case when there are just three particles. In this case, we show in Section 2 that the first collision time corresponds to the first exit time for Brownian motion in a cone (wedge) in $\mathbb{R}^2$, and we can therefore apply the results of Spitzer (1958), and Dante DeBlassie (1987). In Section 3 we consider the special case when the three processes have equal variance. This assumption simplifies the problem greatly and allows for a more elegant approach, from which we obtain the distribution of τ in closed form. This approach also has the attraction that it can be generalised easily and applied to a larger class of Markov processes, where we are interested in the probability of collision. In Section 4, we deduce (in closed form) the first exit time distribution for Brownian motion in a cone of angle $\frac{1}{3}\pi$ in $\mathbb{R}^2$, and we conclude that for the 3-process problem, the restriction imposed on the variances in Section 3 can be weakened. In Section 5 we consider the general problem of n Wiener processes with equal variances, and derive an asymptotic result for the first collision time with the help of the trick used in Section 3.

2. First collision time for three Wiener processes

Consider three independent one-dimensional zero-drift Wiener processes, $\{X_t^i \mid t \geq 0\}_{i=1}^3$, with infinitesimal variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$, respectively, and suppose

$$X_0^1 - X_0^2 = a_1 > 0,$$

$$X_0^2 - X_0^3 = a_2 > 0.$$

Define the first collision time, τ, by

$$\tau = \inf\{s > 0 \mid X_s^i - X_s^j = 0, \text{ some } i \neq j\}. \tag{1}$$

We wish to calculate the distribution of τ. In this section, we show that τ corresponds to the first exit time for Brownian motion in a cone in $\mathbb{R}^2$. Define a new process

(X, Y) by

$$X = \frac{X^1 - X^2 - a_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}, \qquad Y = \frac{X^2 - X^3 - a_2}{\sqrt{\sigma_2^2 + \sigma_3^2}}.$$

Then

$$\tau = \inf\{s > 0 \,|\, (X_s, Y_s) \in \partial A\},$$

where

$$A = \left\{ (x, y) \,\middle|\, x > -\frac{a_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}, y > -\frac{a_2}{\sqrt{\sigma_2^2 + \sigma_3^2}} \right\}.$$

Now if we define the process (B^1, B^2) by

$$B^1 = \frac{X - Y}{S_1} + \frac{1}{S_1}\left(\frac{a_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} - \frac{a_2}{\sqrt{\sigma_2^2 + \sigma_3^2}} \right),$$

$$B^2 = \frac{X + Y}{S_2} + \frac{1}{S_2}\left(\frac{a_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} + \frac{a_2}{\sqrt{\sigma_2^2 + \sigma_3^2}} \right),$$

where $S_1 = SD(X_1 - Y_1)$ and $S_2 = SD(X_1 + Y_1)$, then $B \equiv (B^1, B^2)$ is a Brownian motion in $\mathbb{R}^2$, with

$$B_0 = \left(\frac{1}{S_1}\left(\frac{a_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} - \frac{a_2}{\sqrt{\sigma_2^2 + \sigma_3^2}} \right), \frac{1}{S_2}\left(\frac{a_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} + \frac{a_2}{\sqrt{\sigma_2^2 + \sigma_3^2}} \right) \right). \tag{2}$$

Now

$$\tau = \inf\{s > 0 \,|\, B_s \in \partial C(m)\}, \tag{3}$$

where $C(m)$ is the upright cone in $\mathbb{R}^2$ defined by

$$C(m) = \{(x, y) \in \mathbb{R}^2 \,|\, y > m|x|\}, \tag{4}$$

and

$$m \equiv m(\sigma_1^2, \sigma_2^2, \sigma_3^2) = S_1/S_2. \tag{5}$$

Note that

$$S_1^2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} + \left(\frac{1}{\sqrt{\sigma_2^2 + \sigma_3^2}} + \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right) \sigma_2^2 + \frac{\sigma_3^2}{\sigma_2^2 + \sigma_3^2}, \tag{6}$$

$$S_2^2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} + \left(\frac{1}{\sqrt{\sigma_2^2 + \sigma_3^2}} - \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right)^2 \sigma_2^2 + \frac{\sigma_3^2}{\sigma_2^2 + \sigma_3^2}. \tag{7}$$

We have therefore reduced the problem to determining the distribution of the first exit time of Brownian motion from a cone in $\mathbb{R}^2$. This is given by the next theorem, due to Spitzer (1958).

For $0<\beta<2\pi$, set

$$F(\beta)=\{(r\cos\theta, r\sin\theta)\in\mathbb{R}^2 \mid r\geqslant 0, 0<\theta<\beta\};$$

and, for $p\leqslant\gamma\leqslant 2\pi$, let $F(\beta,\gamma)$ be $F(\beta)$ rotated through an angle γ about the origin.

Let W be a two-dimensional Brownian motion, with $W_0=(R\cos\alpha, R\sin\alpha)$, where $R>0$, $0<\alpha<2\pi$, and let T be the first exit time of W from the cone $F(\beta)$, where $\beta>\alpha$. Let $u(R,\alpha,\beta,t)=\Pr\{T>t\}$. Then:

Theorem 2.1 (Spitzer, 1958, Section 2). *$u(r,\alpha,\beta,t)$ can be obtained by inverting the integral transform*

$$v(s,\alpha,t)=\sqrt{\frac{2}{\pi s}}\int_0^\infty u(r,\alpha,\beta,t)\,\mathrm{e}^{-r^2/2s}\,\mathrm{d}r,$$

for $s>0$, where

$$v(s,\alpha,t)=\frac{2}{\pi}\tan^{-1}\left(\frac{\sin(\alpha\pi/\beta)}{\sinh((\pi/\beta)\sinh^{-1}\sqrt{t/s})}\right). \qquad \square$$

Theorem 2.2 (Spitzer, 1958, Theorem 2). *For $p>0$, $ET^p<\infty$ if and only if $2p\beta<\pi$. This criterion is independent of the initial position $W_0\in F(\beta)$.* $\square$

Putting things together, we get:

Theorem 2.3. *If τ is the first collision time for $\{X_t^i \mid t\geqslant 0\}_1^3$ given by (1), and if we set*

$$\beta=2\tan^{-1}(1/m), \qquad \alpha=\tan^{-1}(B_0^2/B_0^1)-\tan^{-1}m,$$
$$R=\sqrt{(B_0^1)^2+(B_0^2)^2},$$

where m and B_0 are given by (5) and (2), then

$$\Pr\{\tau>t\}=u(R,\alpha,\beta,t),$$

where u is defined in the statement of Theorem 2.1.

We also have, by Theorem 2.2, that for $p>0$, $E\tau^p<\infty$ if and only if $4p\times\tan^{-1}(1/m)<\pi$. $\square$

Dante DeBlassie (1987) considered first exit times for Brownian motion in a cone, in a more general setting, and from here we learn something about the tail of the first collision time distribution.

Theorem 2.4 (Dante DeBlassie, 1987, Corollary 1.3(a)). *In the notation of Theorem 2.3, for some constant C and for fixed R, α,*

$$\Pr\{\tau\geqslant t\}\sim Ct^{-\pi/(2\beta)},$$

as $t\to\infty$. $\square$

It is interesting to note that Theorem 2.4 agrees with the asymptotic results of Evans (1985, Lemma 4) and Uchiyama (1980, Theorem 1.1), concerning Brownian motion in a cone in $\mathbb{R}^2$.

Note that the transformation of the original problem into the problem of determining the first exit time distribution for Brownian motion in a cone has a converse, to make the analogy complete. More precisely, if B is a Brownian motion in $\mathbb{R}^2$, initially at $B_0 \in C(m)$ for some $m \in [1, \infty)$, and if τ_C is the first exit time, then using equations (2) and (5) we can find values $(a_1, a_2, \sigma_1^2, \sigma_2^2, \sigma_3^2)$ so that the distributions of τ_C and τ are identical. We will make use of this fact later in Section 4.

Some interesting special cases of this analogy are given in the following examples.

Example 2.1. If $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma^2$, then $m = \sqrt{3}$, and

$$B_0 = \left(\frac{a_1 - a_2}{\sigma\sqrt{6}}, \frac{a_1 + a_2}{\sigma\sqrt{2}} \right).$$

Example 2.2. If $\sigma_1^2 = \sigma_3^2$ and $\sigma_2^2 = 0$, then the problem reduces to the trivial case where $m = 1$. If $\sigma_1^2 = \sigma_2^2$ and $\sigma_3^2 = 0$, then $m = 1 + \sqrt{2}$.

Example 2.3. If $\sigma_1^2 \neq \sigma_2^2$ and $\sigma_3^2 = 0$, then

$$m = \frac{\sigma_1^2 + \sigma_2^2 + \sigma_2\sqrt{\sigma_1^2 + \sigma_2^2}}{\sigma_1^2 + \sigma_2^2 - \sigma_2\sqrt{\sigma_1^2 + \sigma_2^2}}. \tag{8}$$

Note that for σ_2 fixed, as $\sigma_1 \to 0$, $m \uparrow \infty$; and for σ_1 fixed, as $\sigma_2 \to 0$, $m \downarrow 1$. Thus, since m is continuous in both variables on $(1, \infty)$, given any $m \in (1, \infty)$ we can choose (σ_1, σ_2) so that (10) is satisfied.

The analogy described in this section can be extended easily (but with a great degree of algebraic tedium!) to higher dimensions. The first collision time for n independent Wiener processes can be represented as the first exit time for Brownian motion in a cone in $\mathbb{R}^{n-1}$, and the distribution of this first exit time was calculated by Dante DeBlassie (1987). (Note, however, that to apply the results of Dante DeBlassie there are certain regularity conditions for the cone that need to be checked.)

3. First collision time for three Wiener processes with equal variance

In this section we consider the special case when the three processes have equal variance. This simplifies the problem greatly, and allows for a more elementary approach, yielding nice, closed-form results. The results of this section will be applied in Section 5 when we consider the general problem of determining the first collision time distribution for n independent Wiener processes.

Let $\{X_t \mid t \geqslant 0\}$ be a one-dimensional Wiener process with zero drift, infinitesimal variance σ^2, and $X_0 = 0$. Then for $x, t > 0$,

$$\Pr\left\{\max_{0<s\leqslant t} X_s \geqslant x\right\} = 2 \Pr\{X_t \geqslant x\} = 2\left\{1 - \Phi\left(\frac{x}{\sigma\sqrt{t}}\right)\right\}, \tag{9}$$

where $\Phi(\cdot)$ is the standard normal distribution function. Define

$$\gamma(y) = 2\{1 - \Phi(y)\}. \tag{10}$$

Now consider three independent one-dimensional zero-drift Wiener processes, $\{X_t^i \mid t \geqslant 0\}_{i=1}^3$, each with infinitesimal variance σ^2, and suppose

$$X_0^1 - X_0^2 = a_1 > 0, \qquad X_0^2 - X_0^3 = a_2 > 0.$$

Let τ be defined as before, by (1), and let

$$\tau_{ij} = \inf\{s > 0 \mid X_s^i - X_s^j = 0\} \quad \text{for } i \neq j. \tag{11}$$

Then, by (9),

$$\Pr\{\tau_{12} \leqslant t\} = \gamma\left(\frac{a_1}{\sigma\sqrt{2t}}\right), \tag{12}$$

$$\Pr\{\tau_{23} \leqslant t\} = \gamma\left(\frac{a_2}{\sigma\sqrt{2t}}\right), \tag{13}$$

$$\Pr\{\tau_{13} \leqslant t\} = \gamma\left(\frac{a_1 + a_2}{\sigma\sqrt{2t}}\right). \tag{14}$$

The probability of a collision between any pair of these processes in the time interval $(0, t]$ is given by the following theorem.

Theorem 3.1.

$$\Pr\{\tau \leqslant t\} = \gamma\left(\frac{a_1}{\sigma\sqrt{2t}}\right) + \gamma\left(\frac{a_2}{\sigma\sqrt{2t}}\right) - \gamma\left(\frac{a_1 + a_2}{\sigma\sqrt{2t}}\right).$$

Proof.

$$\begin{aligned} \Pr\{\tau \leqslant t\} &= \Pr \bigcup_{i \neq j} \{\tau_{ij} \leqslant t\} \\ &= \Pr\{(\tau_{12} \leqslant t) \cup (\tau_{23} \leqslant t)\} \\ &= \Pr\{\tau_{12} \leqslant t\} + \Pr\{\tau_{23} \leqslant t\} - \Pr\{\tau_{12} \leqslant t, \tau_{23} \leqslant t\}. \end{aligned} \tag{15}$$

We now show that

$$\Pr\{\tau_{12} \leqslant t, \tau_{23} \leqslant t\} = \Pr\{\tau_{13} \leqslant t\}.$$

Let $E_{ij}=\{\tau_{ij}\leqslant t\}$, and notice that $\tau_{12}\wedge\tau_{23}\leqslant\tau_{13}$. Then

$$\begin{aligned}\Pr\{\tau_{12}\leqslant t,\ \tau_{23}\leqslant t\} &= \Pr\{E_{12},E_{23}\}\\ &= \Pr\{E_{12},E_{23};\ \tau=\tau_{12}\}+\Pr\{E_{12},E_{23};\ \tau=\tau_{23}\}\\ &= \Pr\{E_{12},E_{13};\ \tau=\tau_{12}\}+\Pr\{E_{23},E_{13};\ \tau=\tau_{23}\}\\ &= \Pr\{E_{13};\ \tau=\tau_{12}\}+\Pr\{E_{13};\ \tau=\tau_{23}\}\\ &= \Pr\{E_{13}\}\\ &= \Pr\{\tau_{13}\leqslant t\}.\end{aligned}$$

Thus, (15) becomes

$$\Pr\{\tau\leqslant t\}=\gamma\left(\frac{a_1}{\sigma\sqrt{2t}}\right)+\gamma\left(\frac{a_2}{\sigma\sqrt{2t}}\right)-\gamma\left(\frac{a_1+a_2}{\sigma\sqrt{2t}}\right),$$

as required. □

Remark. It is also possible to do a 'coupling' type sample path proof of Theorem 3.1. By defining slightly modified collision times you can observe that the events in question are actually equal, hence their probabilities are equal.

Corollary 3.2. (i) *For fixed* a_1, a_2, σ^2,

$$\Pr\{\tau>t\}\sim\frac{a_1^2a_2+a_1a_2^2}{4\sqrt{\pi}\sigma^3}\,t^{-3/2},$$

as $t\to\infty$.

(ii) *If* $a_1=a_2=a$, *then for fixed* t, σ^2,

$$\Pr\{\tau>t\}\sim\frac{1}{2\sqrt{\pi}}\,\sigma^{-3}t^{-3/2}a^3,$$

as $a\to 0$.

(iii) (i) *implies that for* $p>0$, $E\tau^p<\infty$ *if and only if* $p<\frac{3}{2}$.

(iv) *For fixed* a_1, a_2, σ^2,

$$\Pr\{\tau>t\}$$
$$\sim 1-2\sigma\sqrt{\frac{t}{\pi}}\left(\frac{1}{a_1}\,\mathrm{e}^{-a_1^2/(4\sigma^2t)}+\frac{1}{a_2}\,\mathrm{e}^{-a_2^2/(4\sigma^2t)}-\frac{1}{a_1+a_2}\,\mathrm{e}^{-(a_1+a_2)^2/(4\sigma^2t)}\right),$$

as $t\to 0$. □

Note that Corollary 3.2(iii) agrees with the result obtained from Theorem 2.2 in Section 2. In this case, $\beta=\frac{1}{3}\pi$. Corollary 3.2(iv) follows from the well-known order relation:

$$\int_x^\infty \mathrm{e}^{-u^2/2}\,\mathrm{d}u\sim\frac{1}{x}\,\mathrm{e}^{-x^2/2}\quad\text{as } x\to+\infty.$$

It is worth noting that the approach taken in this section does not require that the processes be Wiener. In fact, we can easily modify the results to determine the first collision time distribution for three independent Markov processes of the same type, with the same parameters, but of course each having different initial value. This can be done as long as the distribution of the first hitting time (of a single point barrier) is known for the individual processes and their differences, and provided the processes are 'continuous' in the sense that they cannot jump over each other. For example, we can consider a trio of independent one-dimensional Markov diffusions with the same infinitesimal drift and variance. This extension may provide some useful applications.

4. First exit time of Brownian motion from the cone $F(\frac{1}{3}\pi)$

In this section we make use of the analogy described in Section 2 and the results of the previous section to obtain an explicit form for the distribution of the first exit time for Brownian motion in the cone $F(\frac{1}{3}\pi)$. We can then in turn deduce explicit results for the various first collision problems corresponding to this specific cone.

The problem considered in the previous section can be translated into the problem of determining the first exit distribution of a Brownian motion from the cone $C(\sqrt{3})$ in $\mathbb{R}^2$, using the analogy described in Section 2. $C(\sqrt{3})$ is just a rotation of the cone $F(\frac{1}{3}\pi)$. Given that a Brownian motion is initially at $(x, y) \in C(\sqrt{3})$, the distribution of the first exit time τ is given by

$$\Pr\{\tau \leq t\} = \gamma\left(\frac{b_1}{\sqrt{2t}}\right) + \gamma\left(\frac{b_2}{\sqrt{2t}}\right) - \gamma\left(\frac{b_1 + b_2}{\sqrt{2t}}\right), \tag{16}$$

where

$$b_1 = \tfrac{1}{2}\sqrt{6}(x + y/\sqrt{3}), \tag{17}$$

$$b_2 = \tfrac{1}{2}\sqrt{6}(y/\sqrt{3} - x), \tag{18}$$

and γ is defined by (10).

Remark. It seems difficult (if not impossible) to extend this approach further to obtain explicit results for other cones.

We now turn our attention to the set of 3-process collision problems that correspond to this specific cone, $F(\frac{1}{3}\pi)$, thus weakening the restriction imposed on the variances in Section 3.

Consider the three independent Wiener processes introduced in Section 2, and suppose

$$m(\sigma_1^2, \sigma_2^2, \sigma_3^2) = \sqrt{3}, \tag{19}$$

where m is defined by (5). Then, from the results of Section 2, the distribution of the first collision time is the same as the distribution of the first exit time for Brownian motion in $C(\sqrt{3})$, which we have just determined in closed form. Thus, we have a closed-form solution for the distribution of the first collision time, as long as the variances of the processes satisfy (19).

It can be easily checked that the surface defined by (19) is given by

$$S=\{(x,y,z)\in\mathbb{R}^3 \mid 4y^2=(x+y)(y+z);\ x,y,z>0\} \tag{20}$$

$$=\{(x,y,z)\in\mathbb{R}^3 \mid (x/y, y/z)\in L;\ x,y,z>0\}, \tag{21}$$

where

$$L=\left\{(p,q)\in\mathbb{R}^2 \,\middle|\, q=\frac{p+1}{3-p};\ 0<p<3\right\}. \tag{22}$$

5. First collision time for n independent Wiener processes with equal variance

Consider n independent one-dimensional zero-drift Wiener processes $\{X^i\}_1^n$, each with infinitesimal variance σ^2, and suppose that for $i=1,\dots,n-1$, $X_0^i-X_0^{i+1}=a>0$. Consider the events:

$$E_i(t)=\{X_s^i=X_s^{i+1}, \text{some } s<t\}, \qquad E(t)=\bigcup_{i=1}^{n-1} E_i(t).$$

Now $E_i(t)$ is the event of a collision occurring between X^i and X^{i+1} in the time interval $(0,t]$, and $E(t)$ is the event of a collision occurring between *any* pair of these processes in the time interval $(0,t]$.

By Theorem 3.1, we have for each i,

$$PE_i(t)\cup E_{i+1}(t)=2\gamma(d)-\gamma(2d),$$

where $d=a/\sqrt{2\sigma^2 t}$, and γ is defined by (10).

We will make use of the following lemma to obtain an approximation for $PE(t)$ when d is large.

Lemma 5.1. *As $d\to\infty$, $\gamma(2d)/\gamma^3(d)\to 0$.* □

The proof is straightforward, using only the properties of the normal distribution function and l'Hôpital's rule. The main result of this section is:

Proposition 5.2. *For fixed n, as $d\to\infty$,*

$$PE(t)=(n-1)\gamma(d)-\tfrac{1}{2}(n-2)(n-3)\gamma^2(d)+o(\gamma^3(d)).$$

Proof. For convenience we write E for $E(t)$, and E_i for $E_i(t)$. By the principle of inclusion/exclusion, symmetry, independence and Theorem 3.1, we have:

$$\begin{aligned} PE &= P \bigcup_{i=1}^{n-1} E_i \\ &= \sum_i PE_i - \sum_{i<j} P(E_iE_j) + \sum_{i<j<k} P(E_iE_jE_k) - \cdots \\ &= (n-1)\gamma(d) - (n-2)\gamma(2d) - \tfrac{1}{2}(n-2)(n-3)\gamma^2(d) \\ &\quad + \mathrm{o}(\gamma^3(d), \gamma(2d)\gamma(d), P(E_1E_2E_3)). \end{aligned}$$

By Lemma 5.1, $\gamma(2d) = \mathrm{o}(\gamma^3(d))$, and thus $\gamma(2d)\gamma(d) = \mathrm{o}(\gamma^3(d))$ also. And clearly,

$$P(E_1E_2E_3) \leqslant P(E_1E_2) = \gamma(2d).$$

Thus,

$$PE(t) = (n-1)\gamma(d) - \tfrac{1}{2}(n-2)(n-3)\gamma^2(d) + \mathrm{o}(\gamma^3(d)),$$

as required. □

Corollary 5.3. *For fixed n, as $d \to \infty$,*

$$PE(t) \sim (n-1)\gamma(d) \sim \sqrt{\frac{2}{\pi}}\,\frac{n-1}{d}\,\mathrm{e}^{-d^2/2}. \qquad \square$$

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