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Brownian analogues of Burke's theorem

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Abstract

We discuss Brownian analogues of a celebrated theorem, due to Burke, which states that the output of a (stable, stationary) M/M/1 queue is Poisson, and the related notion of quasireversibility. A direct analogue of Burke's theorem for the Brownian queue was stated and proved by Harrison (Brownian Motion and Stochastic Flow Systems, Wiley, New York, 1985). We present several different proofs of this and related results. We also present an analogous result for geometric functionals of Brownian motion. By considering series of queues in tandem, these theorems can be applied to a certain class of directed percolation and directed polymer models. It was recently discovered that there is a connection between this directed percolation model and the GUE random matrix ensemble. We extend and give a direct proof of this connection in the two-dimensional case. In all of the above, reversibility plays a key role. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction and summary

Burke's theorem states that the output of a (stable, stationary) M/M/1 queue (that is, a single-server first-come-first-served queue with Poisson arrivals and exponential holding times—see Section 2 below for a precise definition) is Poisson. This fact was anticipated but not proved by O'Brien (1954) and Morse (1955). Burke (1956) also proved that the output up to a given instant is independent of the number of customers in the queue at that instant. This property is also called *quasi-reversibility*. Discussions on Burke's theorem and related material can be found in the books of Asmussen (1987), Brémaud (1981), Kelly (1979) and Robert (2000).

In this paper we discuss Brownian analogues of Burke's theorem and the related notion of quasi-reversibility. The first can be obtained by taking a 'heavy-traffic' limit of M/M/1 queues, taking care to keep a distinction between the arrivals and service processes. Heavy-traffic queueing models are well-understood (see, for example, Harrison, 1985 or Norros and Salminen, 2000; Williams, 1996 for recent surveys) and in fact this variant of Burke's theorem was presented in Harrison and Williams (1990) in

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an attempt to understand quasi-reversibility in the heavy-traffic context. The resulting queueing system is characterised as follows. We call B a *standard Brownian motion indexed by* $\mathbb{R}$ if $B_0=0$, and $\{B_{-t},\ t\geqslant 0\}$ and $\{B_t,\ t\geqslant 0\}$ are independent standard Brownian motions indexed by $\mathbb{R}_+$. Let B and C be two independent standard Brownian motions indexed by $\mathbb{R}$, and write

$$B_{(s,t)}=B_t-B_s, \qquad C_{(s,t)}=C_t-C_s.$$

Fix $m>0$ and, for $t\in\mathbb{R}$, set

$$q(t)=\sup_{-\infty<s\leqslant t}\{B_{(s,t)}+C_{(s,t)}-m(t-s)\} \tag{1}$$

and

$$d(t)=B_t+q(0)-q(t). \tag{2}$$

Note that q and d are defined on $\mathbb{R}$. The Brownian motion B_t, and Brownian motion with drift $mt-C_t$, can be thought of, respectively, as the *arrivals* and *service* processes, q as the *queue-length* process and d as the *output*, or *departure*, process. We shall refer to the above system as the *Brownian queue.*

We remark that this is slightly different to the usual heavy-traffic set-up (in Harrison, 1985 for example) where $q(0)$ is just given with some distribution and $q(t)$ for $t\geqslant 0$ is defined by

$$q(t)=q(0)+B_t+C_t-mt+M(t), \tag{3}$$

where

$$M(t)=\left(\sup_{0\leqslant s\leqslant t}(-q(0)-B_s-C_s+ms)\right)^+. \tag{4}$$

It can be readily verified that the queue-length process q defined by (1) satisfies (3) and (4) with

$$q(0)=\sup_{-\infty<s\leqslant 0}\{B_{(s,0)}+C_{(s,0)}+ms\} \tag{5}$$

and that $\{q(t),\ t\geqslant 0\}$ is stationary for this choice of $q(0)$. (See, for example, the survey of Norros and Salminen, 2000.)

The analogue of Burke's theorem in this context is that d is a standard Brownian motion indexed by $\mathbb{R}$ and, moreover, the values of d up to a given instant t are independent of $q(t)$. An equivalent result was presented in Harrison and Williams (1990). As was observed there, it is closely related to Pitman's representation of the three-dimensional Bessel process (Pitman, 1975) and Williams' path decomposition of Brownian motion (Williams, 1974), as extended in Pitman and Rogers (1981) to the case of non-zero drift. This will be discussed in detail in Section 2.

Pitman's representation theorem (Pitman, 1975), which we will encounter throughout this paper, together with a number of variants, states that for $\{B_u,\ u\geqslant 0\}$ a standard Brownian motion, $\{2(\sup_{s\leqslant u}B_s)-B_u,\ u\geqslant 0\}$ is distributed as the norm of a three-dimensional Brownian motion. Matsumoto and Yor (1999, 2001) have recently obtained a version of Pitman's representation theorem for geometric Brownian motions, and a variety of related results, which essentially rely upon the observation that, if one replaces 'sup' by '$\log\int\exp$' in the statement of Pitman's theorem, the result holds true with the three-dimensional Bessel process replaced by another Markov process. This can be thought of as a generalisation because Pitman's theorem can be recovered by rescaling and applying Laplace's method. It turns out that, if one replaces 'sup' by '$\log\int\exp$' in definition (1) of the Brownian queue, the conclusion remains valid: d is a standard Brownian motion, and $\{d(s),\ s\leqslant t\}$ is independent of $q(t)$. This follows from results presented in Matsumoto and Yor (2001). We refer to this as the *generalised Brownian queue*. This is the topic of Section 3.

In Section 4, by considering a sequence of n Brownian queues in tandem, we are led to a variational formula which relates the 'total occupancy' of the system to the process

$$L_n(t)=\sup_{0\leqslant s_1\leqslant\cdots\leqslant s_{n-1}\leqslant t}\{B^{(1)}_{(0,s_1)}+\cdots+B^{(n)}_{(s_{n-1},t)}\}, \tag{6}$$

where $B^{(1)},B^{(2)},\ldots$ is a sequence of independent standard Brownian motions. Related formulas have appeared before in Muth (1979), Szczotka and Kelly (1990), Ganesh (1998). We describe how this variational formula can be used, following a program introduced by Seppäläinen (1998) (see O'Connell, 1999 for a survey), to show that, $L_n(1)/\sqrt{n}\to 2$ almost surely, as $n\to\infty$. We remark that this limiting result can also be deduced from the recent observation, independently made by Baryshnikov (2001) and Gravner et al. (2000), that $L_n(1)$ has the same law as the largest eigenvalue of a GUE random matrix of order n. (GUE stands for 'Gaussian Unitary Ensemble'.)

Related work on heavy-traffic queues in tandem is presented in Glynn and Whitt (1991) and Harrison and Williams (1992). In fact, the processes L_n were introduced in Glynn and Whitt (1991), albeit with a slightly different interpretation; Harrison and Williams (1992) present quasireversibility results in the more general context of 'feedforward queueing networks'.

In Section 5, we consider a sequence of generalised Brownian queues in tandem. This leads to a variety of large deviations results which can be interpreted in terms of a certain random polymer in a random medium. For example, we can compute the free energy density

$$f(\beta)=\lim_{n\to\infty}\frac{1}{n}\log Z_n(\beta),$$

where

$$Z_n(\beta)=\int_{0<s_1<\cdots<s_{n-1}<n}\mathrm{d}s_1\ldots\mathrm{d}s_{n-1}\exp\{\beta(B^{(1)}_{(0,s_1)}+\cdots+B^{(n)}_{(s_{n-1},n)})\}.$$

In Section 6 we make some remarks concerning the identity in law between $L_n(1)$ and the largest eigenvalue of a GUE random matrix of order n. We show how, combined with a recent observation of Johansson (2000), this yields a certain asymptotic relationship between the Laguerre and Wigner ensembles. We also give a direct proof of the identity in the case $n=2$, and show that in fact there is a process-version which is closely related to Pitman's representation theorem. Further connections between this identity in law and Burke's theorem are presented in O'Connell and Yor (2001). See also König et al. (2001) for connections with discrete orthogonal polynomial ensembles, and Hambly et al. (2001b) for related results in a non-Markovian setting.

In Section 7, we present a general result which states that the measure-preserving property of a certain path-transformation is equivalent to the reversibility of another; this demonstrates the key role played by reversibility, and provides an alternative method for proving the Burke-type theorems presented in earlier sections.

In Section 8 we present some multi-dimensional extensions.

2. Burke's theorem and the Brownian queue

In this section we recall and extend a special case of the quasi-reversibility result for Brownian queueing models which was presented in Harrison and Williams (1990).

Let A and S be independent Poisson processes on the real line with respective intensities $0 < \lambda < \mu$. Then the process Q, defined for $t \in \mathbb{R}$ by

$$Q(t) = \sup_{-\infty < s \leqslant t} \{A(s,t] - S(s,t]\}^+$$

is a stationary (and reversible) birth and death process. Here, $A(s,t]$ is the Poisson measure induced by A of the half-open interval $(s,t]$. This is the classical M/M/1 queue: A is the *arrivals process*, S is the *service process* and Q is the *queue-length process*. The *departure process* D is defined for $-\infty < s < t < \infty$ by

$$D(s,t] = A(s,t] + Q(s) - Q(t).$$

The M/M/1 queue has the following remarkable property, which follows from the fact that the process Q is reversible (see, for example, Kelly, 1979; Robert, 2000). This observation is originally due to Burke (1956).

Theorem 1. (1) *D is a Poisson process with intensity λ.*
(2) *$\{D(s,t],\ s \leqslant t\}$ is independent of $\{Q(s),\ s \geqslant t\}$.*

Letting λ and μ tend to infinity in the right way, we obtain the following analogue of Theorem 1 for Brownian motions. Let B and C be two independent standard Brownian motions indexed by the entire real line, and write

$$B_{(s,t)} = B_t - B_s, \qquad C_{(s,t)} = C_t - C_s.$$

Fix $m > 0$ and, for $t \in \mathbb{R}$, set

$$q(t) = \sup_{-\infty < s \leqslant t} \{B_{(s,t)} + C_{(s,t)} - m(t-s)\} \tag{7}$$

and

$$d(t) = B_t + q(0) - q(t). \tag{8}$$

We shall refer to this system as the *Brownian queue*. The following is a special case of a result presented in Harrison and Williams (1990).

Theorem 2. (1) *$\{d(t),\ t \in \mathbb{R}\}$ is a standard Brownian motion (indexed by $\mathbb{R}$).*
(2) *For each $t \in \mathbb{R}$, $\{d(s),\ s \leqslant t\}$ is independent of $\{q(s),\ s \geqslant t\}$.*

Theorem 2 can be obtained from Theorem 1 by weak convergence arguments: if $\mu \to \infty$ and $(\mu - \lambda)/\sqrt{\mu} \to m$, then (a sufficiently refined version of) Donsker's theorem yields the result. In Harrison and Williams (1990) a more general result is proved (in a 'multiclass' context) using time-reversal arguments, but they also give a proof which demonstrates the connection with Pitman's representation theorem and Williams' path decomposition of Brownian motion. We present a variant of that proof here.

We remark that, in the Brownian storage model discussed in Norros and Salminen (2000), the *storage* process is defined by (7) but the output process is defined by

$$\hat{d}(t) = B_t + C_t + q(0) - q(t). \tag{9}$$

Unlike d, the process $\hat{d}$ is *not* a Brownian motion; its law is characterised in Norros and Salminen (2000) Let $B^{(\mu)}$ be a standard Brownian motion with drift $\mu > 0$, indexed by $\mathbb{R}$. For $t \geqslant 0$, set

$$\rho_t = \sup_{-\infty < s \leqslant t} \{B_s^{(\mu)} - B_t^{(\mu)}\} \tag{10}$$

and

$$\hat{B}_t^{(\mu)} = B_t^{(\mu)} + 2(\rho_t - \rho_0). \tag{11}$$

The following result was presented in Harrison and Williams (1990). For completeness, we include a proof.

Theorem 3. (1) *The process* $\{\hat{B}^{(\mu)},\ t \geqslant 0\}$ *is a standard Brownian motion with drift* μ.
(2) $\{\hat{B}_s^{(\mu)},\ 0 \leqslant s \leqslant t\}$ *is independent of* $\{\rho_s,\ s \geqslant t\}$.

Proof. (1) If we set $X^{(\mu)} = \sup_{s<0} B_s^{(\mu)}$ and $M_t^{(\mu)} = \sup_{0<s<t} B_s^{(\mu)}$, then

$$\hat{B}_t^{(\mu)} = 2(M_t^{(\mu)} - X^{(\mu)})^+ - B_t^{(\mu)}$$

and $X^{(\mu)}$ is exponentially distributed with mean $1/2\mu$. An extension of Pitman's representation (Pitman, 1975) for the three-dimensional Bessel process, obtained in Pitman and Rogers (1981), states that the process $2M^{(\mu)} - B^{(\mu)}$ is autonomously Markov; the law of this process is denoted by $\mathrm{BES}^3(0,\mu)$ (see, for example, Pitman and Rogers, 1981). Now recall the following path decomposition of Brownian motion with positive drift about its infimum, which extends Williams' path decomposition of Brownian motion. This is also presented in Pitman and Rogers (1981). Let $B^{(-\mu)}$ be a standard Brownian motion with negative drift $-\mu$, R a $\mathrm{BES}^3(0,\mu)$ and Z an exponential random variable with mean $1/2\mu$, all independent of each other. Set

$$\tau = \inf\{t \geqslant 0\colon B_t^{(-\mu)} = -Z\}.$$

Then the process Y defined by

$$Y_t = \begin{cases} B_t^{(-\mu)}, & t \leqslant \tau, \\ R_{t-\tau} - Z, & t \geqslant \tau \end{cases}$$

is standard Brownian motion with drift μ. Combining these two facts, we see that $\hat{B}^{(\mu)}$ is standard Brownian motion with drift μ, as required.

(2) follows from the formula

$$\rho_t = \sup_{s \geqslant t} (\hat{B}_t^{(\mu)} - \hat{B}_s^{(\mu)}). \tag{12}$$

To see that this formula is valid, write

$$\sup_{s \geqslant t} (\hat{B}_t^{(\mu)} - \hat{B}_s^{(\mu)}) = 2S_t^{(\mu)} - B_t^{(\mu)} - \inf_{s \geqslant t} (2S_s^{(\mu)} - B_s^{(\mu)}),$$

where $S_t^{(\mu)} = \sup_{s \leqslant t} B_s^{(\mu)}$, and observe that

$$\inf_{s \geqslant t} (2S_s^{(\mu)} - B_s^{(\mu)}) = S_t^{(\mu)}. \tag{13}$$

Thus

$$\sup_{s \geqslant t} (\hat{B}_t^{(\mu)} - \hat{B}_s^{(\mu)}) = S_t^{(\mu)} - B_t^{(\mu)} = \rho_t,$$

as required. The identity (13) is at the heart of Pitman's representation theorem, although it is usually stated with $S^{(\mu)}$ replaced by $M^{(\mu)}$ (the maximum over a finite time-interval). □

Proof of Theorem 2. Define two independent Brownian motions

$$\beta^{(1)} = \frac{B - C}{\sqrt{2}}, \qquad \beta^{(2)} = \frac{B + C}{\sqrt{2}}.$$

Now we can write, for $t \geqslant 0$,

$$d(t) = \frac{1}{\sqrt{2}} \beta_t^{(1)} + \frac{1}{\sqrt{2}} \{-\beta_t^{(2)} + 2vt - 2(S_t^{(v)} - X^{(v)})^+\},$$

where $v = m/\sqrt{2}$,

$$S_t^{(v)} = \sup_{0 \leqslant s \leqslant t} (-\beta_s^{(2)} + vs)$$

and

$$X^{(v)} = \sup_{-\infty < s \leqslant 0} (-\beta_s^{(2)} + vs).$$

Set $\gamma_t^{(v)} = -\beta_t^{(2)} + vt$. We also have, for $t \geqslant 0$,

$$\rho_t := \frac{1}{\sqrt{2}} q(t) = \max(S_t^{(v)}, X^{(v)}) - \gamma_t^{(v)}.$$

Now, by Theorem 3,

(1) The process defined, for $t \geqslant 0$, by

$$a_t = 2(S_t^{(v)} - X^{(v)})^+ - \gamma_t^{(v)}$$

is a Brownian motion with drift v, and

(2) for each $t \geqslant 0$, $\{a_s,\ 0 \leqslant s \leqslant t\}$ is independent of $\{\rho_s,\ s \geqslant t\}$.

Since $\beta^{(1)}$ and $\beta^{(2)}$ are independent, the statement of Theorem 2, with time-indices restricted to $t \geqslant 0$, follows. To extend it to $t \in \mathbb{R}$, note that by a simple translation of the time origin, the above argument shows that, for any $s < 0$, the process $\{d_{(s,t)},\ t \geqslant s\}$ is a standard Brownian motion and for any fixed $t > s$, $\{d_{(s,r)},\ s \leqslant r \leqslant t\}$ is independent of $\{q(u),\ u \geqslant t\}$. Here we are using the notation $d_{(s,t)} = d(t) - d(s)$. By time-reversal on the interval $[s, 0]$, this implies that $\{d(-u),\ 0 \leqslant u \leqslant -s\}$ is a standard Brownian motion indexed by $[0, -s]$, for any $s < 0$. Here we are using the elementary fact that, if B is a standard Brownian motion indexed by an interval $[0, T]$ with $B(0) = 0$, then so is $\{B_{T-t} - B_T,\ 0 \leqslant t \leqslant T\}$. Thus, $\{d(-t),\ t \geqslant 0\}$ and $\{d(t),\ t \geqslant 0\}$ are two independent standard Brownian motions indexed by $\mathbb{R}_+$ and, for each t, $\{d(s),\ s \leqslant t\}$ is independent of $\{q(u),\ u \geqslant t\}$, as required. $\square$

Note that if we define

$$e(t) = C_t + q(0) - q(t)$$

then (for $t \geqslant 0$)

$$e(t) = -\frac{1}{\sqrt{2}}\beta_t^{(1)} + \frac{1}{\sqrt{2}}\{-\beta_t^{(2)} + 2vt - 2(S_t^{(v)} - X^{(v)})^+\}$$

and we immediately obtain the following extension of Theorem 2. (The extension from $t \geqslant 0$ to $t \in \mathbb{R}$ can be carried out as in the proof of Theorem 2 above.)

Theorem 4. (1) *d and e are independent standard Brownian motions indexed by $\mathbb{R}$.* (2) *$\{(d(s), e(s)) - \infty < s \leqslant t\}$ is independent of $\{q(u),\ u \geqslant t\}$.*

Theorem 4 is an easy but significant extension of Theorem 2; in the Poisson case, the analogue of Theorem 4 holds—this is presented in O' Connell and Yor (2001) and used to obtain a representation for independent Poisson processes *conditioned never to collide* as a functional of (unconditioned) independent Poisson processes. It is also closely related to the connection between queues and random matrices mentioned in Section 1; we will discuss this in more detail in Section 6.

In Section 4 we will relate the results of this section to a continuous directed percolation problem, by considering a series of Brownian queues in tandem.

3. The generalised Brownian queue

In this section we will define a *generalised Brownian queue*, with the 'sup' in (7) of Section 2 replaced by '$\log \int \exp$', and show that the corresponding analogues of Theorems 2, 3 and 4 hold; in fact, they follow directly from results presented in Matsumoto and Yor (2001). We remark that Theorems 2, 3 and 4 can be recovered from the results of this section by rescaling and applying Laplace's method.

These results will be applied in Section 5 to compute the free energy density (logarithm of the partition function) of a certain directed polymer in a random medium and other large deviations results which can be related to the Brownian percolation model of Section 4.

As before, let B and C be independent standard Brownian motions indexed by $\mathbb{R}$, and $m > 0$ be a fixed constant. For $t \in \mathbb{R}$, set

$$r(t) = \log \int_{-\infty}^{t} \mathrm{d}s \exp\{B_{(s,t)} + C_{(s,t)} - m(t-s)\}, \tag{14}$$

$$f(t) = B_t + r(0) - r(t) \tag{15}$$

and

$$g(t) = C_t + r(0) - r(t). \tag{16}$$

Note that $r(t)$ is clearly stationary in t; to see that $r(0) < \infty$ almost surely simply note that, with probability one, $B_{(s,0)} + C_{(s,0)} + ms < ms/2$ for all s sufficiently negative (by Strassen's law of the iterated logarithm, for example). In fact, $r(0)$ has the same law as $-\log Z_m$, where Z_m is gamma-distributed with parameter m: this is Dufresne's identity (Dufresne, 2001). Similar remarks apply to the integrals α_t and A_t defined below.

We shall refer to the above system as the *generalised Brownian queue.*

Theorem 5. (1) *f and g are independent standard Brownian motions indexed by $\mathbb{R}$.*
(2) *For each $t \in \mathbb{R}$, $\{(f(s), g(s)),\ -\infty < s \leqslant t\}$ is independent of $\{r(s),\ s \geqslant t\}$.*

To prove this, we will first write down an analogue of Theorem 3. Let $B^{(\mu)}$ be a standard Brownian motion with drift $\mu > 0$, indexed by the entire real line. For $t \geqslant 0$, set

$$\alpha_t = \log \int_{-\infty}^{t} \mathrm{d}s \exp\{2(B_s^{(\mu)} - B_t^{(\mu)})\} \tag{17}$$

and

$$\hat{B}_t^{(\mu)} = B_t^{(\mu)} + \alpha_t - \alpha_0. \tag{18}$$

Theorem 6. (1) *The process $\{\hat{B}_t^{(\mu)},\ t \geqslant 0\}$ is a standard Brownian motion with drift μ.*
(2) *For each $t \geqslant 0$, $\{\hat{B}_s^{(\mu)},\ 0 \leqslant s \leqslant t\}$ is independent of $\{\alpha_s,\ s \geqslant t\}$.*

Proof. (1) follows from Matsumoto and Yor (2001, Theorem 2.1). (2) follows from the formula

$$\alpha_t = \log \int_{s \geqslant t} \exp\{2(\hat{B}_t^{(\mu)} - \hat{B}_s^{(\mu)})\}. \tag{19}$$

This formula is implicit in Matsumoto and Yor (2001), but since it is not stated explicitly there we present a proof here for completeness. Set

$$A_t = \int_{s \leqslant t} \exp(2B_s^{(\mu)})\, \mathrm{d}s.$$

From the definition of $\hat{B}^{(\mu)}$,

$$\int_{s \geqslant t} \mathrm{d}s \exp\{2(\hat{B}_t^{(\mu)} - \hat{B}_s^{(\mu)})\} = \exp(-2B_t^{(\mu)}) A_t^2 \int_{s \geqslant t} A_s^{-2} \exp(2B_s^{(\mu)})\, \mathrm{d}s$$

and (19) follows by noting that

$$\mathrm{d}A_s^{-1} = -A_s^{-2}\exp(2B_s^{(\mu)})\,\mathrm{d}s,$$

and hence

$$\int_{s\geqslant t} A_s^{-2}\exp(2B_s^{(\mu)})\,\mathrm{d}s = A_t^{-1}. \qquad \square$$

The proof of Theorem 6(2) given in Matsumoto and Yor (2001) uses the method of enlargement; in Section 7 we will give an alternative proof using reversibility arguments.

Proof of Theorem 5. Define two independent Brownian motions

$$\beta^{(1)} = \frac{B-C}{\sqrt{2}}, \qquad \beta^{(2)} = \frac{B+C}{\sqrt{2}}$$

and proceed by applying Theorem 6 as in the proofs of Theorems 2 and 4. Note that α here plays the role of 2ρ in the proof of Theorem 2. $\square$

We conclude this section with some remarks on the process α, which is in fact the logarithm of a particular 'generalised Ornstein-Uhlenbeck process', as discussed in Carmona et al. (2001).

Theorem 7. *The process* α *defined by* (17) *is a stationary*, *reversible Markov process. The Markov semigroup associated with* $\exp(\alpha)$ *has infinitesimal generator*

$$2x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (1+2x(1-\mu))\frac{\mathrm{d}}{\mathrm{d}x}.$$

The stationary distribution of α (*that is, the law of* α_0) *is the law of* $-\log Z_{2\mu}$, *where* $Z_{2\mu}$ *is gamma-distributed with parameter* 2μ.

Proof. The process $\exp(\alpha)$ is stationary by construction, since $B^{(\mu)}$ has stationary increments. The Markov property is argued in Carmona et al. (2001) (this presents no difficulty and holds for any Lévy process in lieu of $B^{(\mu)}$). The reversibility of $\exp(\alpha)$ is proved in Donati-Martin et al. (2001), where the invariant distribution is given explicitly (see the remark below). Alternatively, note that it follows from Theorem 6(1) and the symmetry inherent in the formula (19). The fact that α_0 has the same law as $-\log Z_{2\mu}$ is *Dufresne's identity* (see, for example, Carmona et al., 2001; Dufresne, 2001a,b; Matsumoto and Yor, 1999; Yor, 1992). $\square$

Remark. We note that the proof of reversibility given in Donati-Martin et al. (2001) shows that the semigroup in Theorem 7 is symmetric with respect to

$$\gamma_\mu(\mathrm{d}x) = x^{2\mu-1}\mathrm{e}^{-x}\,\mathrm{d}x,$$

for every $\mu\in\mathbb{R}$; γ_μ is a bounded measure for $\mu>0$, and unbounded for $\mu\leqslant 0$.

4. Brownian queues in tandem and directed percolation

We now turn to a related directed percolation problem. To begin with, we construct a 'tandem of Brownian queues'. Let $B, B^{(1)}, B^{(2)}, \dots$ be a sequence of independent standard Brownian motions, each indexed by $\mathbb{R}$. For $-\infty < s \leqslant t < \infty$, set

$$q_1(t) = \sup_{-\infty < s \leqslant t} \{B_{(s,t)} + B^{(1)}_{(s,t)} - m(t-s)\}, \tag{20}$$

$$d_1(s,t) = B_{(s,t)} + q_1(s) - q_1(t) \tag{21}$$

and for each $k = 2, 3, \dots$ set

$$q_k(t) = \sup_{-\infty < s \leqslant t} \{d_{k-1}(s,t) + B^{(k)}_{(s,t)} - m(t-s)\}, \tag{22}$$

$$d_k(s,t) = d_{k-1}(s,t) + q_k(s) - q_k(t). \tag{23}$$

The queueing interpretation of the above quantities is as follows. The process B is the *arrivals process* at the first queue; $mt - B^{(k)}_t$ is the *service process* at the kth queue; d_k is the *departure process* from the kth queue; q_k is the kth queue-length process.

It follows from Theorem 2 that $q_1(0), q_2(0), \dots$ is a sequence of i.i.d. random variables. (This statement is equivalent to the statement that stationary heavy-traffic tandem networks have product form solutions, as demonstrated in Harrison and Williams, 1992.) The distribution of $q_1(0)$ is exponential with mean $1/m$. Moreover, by construction, we have

$$\sum_{k=1}^{n} q_k(0) = \sup_{t>0} \{B_{(-t,0)} - mt + L_n(t)\}, \tag{24}$$

where

$$L_n(t) = \sup_{0 \leqslant s_1 \leqslant \cdots \leqslant s_{n-1} \leqslant t} \{B^{(1)}_{(-t,-s_{n-1})} + \cdots + B^{(n)}_{(-s_1,0)}\}. \tag{25}$$

To see this, note to begin with that

$$q_2(t) = \sup_{-\infty < s \leqslant t} \{d_1(s,t) + B^{(2)}_{(s,t)} - m(t-s)\}, \tag{26}$$

so by the definitions of d_1 and q_1,

$$\begin{aligned} q_1(0) + q_2(0) &= \sup_{-\infty < s \leqslant 0} \{B(s,0) + q_1(s) + B^{(2)}_{(s,0)} + ms\} \\ &= \sup_{-\infty < r < s \leqslant 0} \{B(s,0) + B(r,s) + B^{(1)}_{(r,s)} - m(s-r) + B^{(2)}_{(s,0)} + ms\} \\ &= \sup_{r<0} \{B(r,0) + mr + L_2(-r)\}, \end{aligned}$$

now proceed by induction on n.

The random variables $L_n(t)$ arise naturally in a continuous *directed percolation* model for fluid-flow through a random medium with directional constraints; the medium in this model is represented by white-noise indexed by $\mathbb{R}_+ \times \mathbb{Z}_+$, specifically $\{dB^{(k)}_t,\ (t,k) \in \mathbb{R}_+ \times \mathbb{Z}_+\}$, and the portion of the medium visited by the fluid at time s is given by the random set

$$\{(t,n) \in \mathbb{R}_+ \times \mathbb{Z}_+ : L_n(t) \leqslant s\}.$$

A discrete version of this model is also known as the 'corner growth model'; here the $L_n(t)$ are replaced by the $D(m,n)$ defined in Section 6. For further discussion regarding the continuous directed percolation model, see Hambly et al. (2001a).

Historical note. Discrete versions of the formula (24) have appeared before in Szczotka and Kelly (1990) and Ganesh (1998); see also Baccelli et al. (2000), where a similar formula is used to prove ergodic properties of infinitely many queues in tandem. The connection between queues in tandem and directed percolation was first reported in Muth (1979). The random processes L_n were introduced in Glynn and Whitt (1991), in a similar context.

It is easy to see, by Kingman's subadditive ergodic theorem, that the limit

$$l(x) = \lim_{n\to\infty} L_n(xn)/n$$

exists almost surely for each $x > 0$. We will now show how (24) can be used to identify the function l. See O' Connell (1999) for a survey on the application of this technique in the context of discrete queueing systems; the main idea originates in Seppäläinen (1998).

First note that, by Brownian scaling, $l(x)$ is proportional to $\sqrt{x}$. To identify the constant of proportionality, we normalise the variational formula (24) and let $n \to \infty$ to obtain (modulo technicalities):

$$1/m = \sup_{x>0} \{-mx + l(x)\}. \tag{27}$$

The formula (27) is valid for any $m > 0$, and is essentially a Legendre transform. Since l is concave, it can be inverted, and we obtain $l(x) = 2\sqrt{x}$. We will not prove this here because it actually follows from a recent observation, due to Baryshnikov (2001) and Gravner et al. (2000), namely that the random variable

$$M_n = \sup_{0\leqslant s_1\leqslant\cdots\leqslant s_{n-1}\leqslant 1} \{B^{(1)}_{(0,s_1)} + \cdots + B^{(n)}_{(s_{n-1},1)}\}, \tag{28}$$

has the same law as the largest eigenvalue of an n-dimensional GUE random matrix. Therefore, in particular,

$$\lim_{n\to\infty} M_n/\sqrt{n} = 2, \tag{29}$$

almost surely. Now, by Brownian scaling, this is equivalent to the statement that $l(x) = 2\sqrt{x}$. A proof of this, which uses the variational formula and also yields sharp uniform concentration, will be presented in Hambly et al. (2001a). We shall return to the random matrix connection in Section 6.

5. Generalised Brownian queues in tandem and directed polymers

In this section we construct a tandem of generalised Brownian queues and apply Theorem 5 to obtain large deviations results related to the directed percolation problem of the previous section. These results can be interpreted in terms of the partition function associated with a certain directed polymer in a random medium.

Let $B, B^{(1)}, B^{(2)}, \ldots$ be a sequence of independent standard Brownian motions, each indexed by $\mathbb{R}$, and let $m > 0$ be a fixed constant. For $-\infty < s \leqslant t < \infty$, set

$$r_1(t) = \log \int_{-\infty}^{t} \mathrm{d}s \exp\{B_{(s,t)} + B^{(1)}_{(s,t)} - m(t-s)\},$$

$$f_1(s,t) = B_{(s,t)} + r_1(s) - r_1(t)$$

and for each $k = 2, 3, \ldots$ set

$$r_k(t) = \log \int_{-\infty}^{t} \mathrm{d}s \exp\{f_{k-1}(s,t) + B^{(k)}_{(s,t)} - m(t-s)\},$$

$$f_k(s,t) = f_{k-1}(s,t) + q_k(s) - q_k(t).$$

It follows from Theorem 5 that $r_1(0), r_2(0), \ldots$ is a sequence of i.i.d. random variables. Moreover, $r_1(0)$ has the same law as $-\log Z_m$, where Z_m is gamma-distributed with parameter m. By construction, we have

$$\sum_{k=1}^{n} r_k(0) = \log \left[\int_{-\infty}^{0} \mathrm{d}u \exp(B_{(u,0)} + mu) \right.$$
$$\left. \times \int_{u < s_1 < \cdots < s_{n-1} < 0} \mathrm{d}s_1 \ldots \mathrm{d}s_{n-1} \exp\{B^{(1)}_{(u,s_1)} + \cdots + B^{(n)}_{(s_{n-1},0)}\} \right]. \quad (30)$$

This is the $\log \int \exp$ analogue of the variational formula (24), and is derived in exactly the same way. Applying the strong law of large numbers, we obtain the following result.

Theorem 8. *We have, for each $m > 0$:*

$$\lim_{n\to\infty} \frac{1}{n} \log \int_{u < s_1 < \cdots < s_{n-1} < 0} \mathrm{d}u\, \mathrm{d}s_1 \ldots \mathrm{d}s_{n-1} \exp\{mu + B^{(1)}_{(u,s_1)} + \cdots + B^{(n)}_{(s_{n-1},0)}\}$$
$$= -\Psi(m) \quad (31)$$

almost surely, where

$$\Psi(m) = E \log Z_m = \Gamma'(m)/\Gamma(m)$$

is the digamma function (and Γ is the Gamma function).

We defer the proof. Theorem 8 can be interpreted as follows. Let $\mathscr{B}$ denote the σ-field generated by the Brownian motions $B^{(1)}, B^{(2)}, \ldots$, and let $\tau_1, \tau_2, \ldots$ be the points of a unit-rate Poisson process on $\mathbb{R}_+$, independent of $\mathscr{B}$. Set

$$E_n = B^{(1)}_{(0,\tau_1)} + \cdots + B^{(n)}_{(\tau_{n-1},\tau_n)}.$$

By Brownian scaling, Theorem 8 is equivalent to:

Theorem 9. *For $\theta \neq 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \log E[\exp(\theta E_n)|\mathscr{B}] = -2\log\theta - \Psi(1/\theta^2), \quad (32)$$

almost surely.

Thus, if we set

$$\Lambda(\theta)=\begin{cases} -2\log\theta-\Psi(1/\theta^2), & \theta\neq 0,\\ 0, & \theta=0, \end{cases} \tag{33}$$

we have

$$\lim_{n\to\infty}\frac{1}{n}\log E[\exp(\theta E_n)|\mathscr{B}]=\Lambda(\theta), \tag{34}$$

almost surely. It is easy to check that Λ is finite and differentiable everywhere with $\Lambda(0)=\Lambda'(0)=0$. (The digamma function is finite and differentiable for positive values of its argument and for large x, $\Psi(x)\sim\log x$ and $\Psi'(x)\sim 1/x$. See, for example, Abramowitz and Stegun, 1970; Lebedev, 1972.) We have therefore obtained a *quenched large deviation principle*, associated with the conditional law of large numbers: given $\mathscr{B}$, $E_n/n\to 0$ almost surely. For example, Theorem 9 implies that, for any $x>0$,

$$\lim_{n\to\infty}\frac{1}{n}\log P(E_n>xn|\mathscr{B})=-\Lambda^*(x),$$

almost surely, where

$$\Lambda^*(x)=\sup_{\theta\in\mathbb{R}}[x\theta-\Lambda(\theta)].$$

(See, for example, Dembo and Zeitouni, 1998.)

Proof of Theorem 9 (*and hence Theorem* 8). By (30),

$$\lim_{n\to\infty}\frac{1}{n}\log E[\exp(\theta(E_n+B_{\tau_n}))|\mathscr{B},B]=\Lambda(\theta), \tag{35}$$

almost surely. We shall use some tools from large deviation theory—see, for example, Dembo and Zeitouni (1998). Since Λ is finite and differentiable everywhere, and satisfies the steepness condition, we see that, conditional on $\mathscr{B}$ and B, the sequence $(E_n+B_{\tau_n})/n$ almost surely satisfies the large deviation principle in $\mathbb{R}_+$ with good convex rate function Λ^*. It therefore suffices to show that, conditional on $\mathscr{B}$ and B, the sequences $(E_n+B_{\tau_n})/n$ and E_n/n are almost surely exponentially equivalent; that is,

$$\limsup_{n\to\infty}\frac{1}{n}\log P(B_{\tau_n}>\delta n|B)=-\infty,$$

almost surely, for any $\delta>0$. This would imply that the sequence E_n/n almost surely satisfies the same large deviation principle and the result will follow from Varadhan's lemma. Fix $a\in(\frac{1}{2},1)$ and set $M=\sup_{t>0}B_t/(1+t^a)$. Then $M<\infty$ almost surely and

$$\limsup_{n\to\infty}\frac{1}{n}\log P(B_{\tau_n}>\delta n|B)\leqslant\limsup_{n\to\infty}\frac{1}{n}\log P(\tau_n^a>\delta nM^{-1}-1|M) \tag{36}$$

$$=-\infty, \tag{37}$$

almost surely, as required. □

The corresponding 'annealed' (unconditional) large deviation principle is easy to compute: for $-\sqrt{2}<\theta<\sqrt{2}$,

$$\lim_{n\to\infty}\frac{1}{n}\log E\exp(\theta E_n)=\log E\exp(\theta^2\tau_1/2)=-\log(1-\theta^2/2). \tag{38}$$

As there are two sources of randomness, there is another quenched large deviation principle which is obtained by conditioning on the σ-field $\mathcal{T} = \sigma(\tau_1, \tau_2, \ldots)$; in this case the rate function is quadratic:

$$\lim_{n\to\infty} \frac{1}{n} \log E[\exp(\theta E_n)|\mathcal{T}] = \theta^2/2. \tag{39}$$

Omitting technical details, we will now show how this relates to a model for a directed polymer in a random medium, and to the directed percolation problem of the previous section. For $x < 0$, the limit

$$\gamma(x) = \lim_{n\to\infty} \frac{1}{n} \log \int_{xn<s_1<\cdots<s_{n-1}<0} \mathrm{d}s_1 \ldots \mathrm{d}s_{n-1} \exp\{B^{(1)}_{(xn,s_1)} + \cdots + B^{(n)}_{(s_{n-1},0)}\}$$

exists almost surely, by Kingman's subadditive ergodic theorem. It is easy to check that γ is a concave function. By Laplace's method (this would require justification),

$$-\Psi(m) = \sup_{x<0} [mx + \gamma(x)] = (-\gamma)^*(m).$$

Thus, by inversion, $\gamma = -(-\Psi)^*$.

For $\beta > 0$, set

$$Z_n(\beta) = \int_{-n<s_1<\cdots<s_{n-1}<0} \mathrm{d}s_1 \ldots \mathrm{d}s_{n-1} \exp\{\beta(B^{(1)}_{(-n,s_1)} + \cdots + B^{(n)}_{(s_{n-1},0)})\}.$$

This can be thought of as a partition function, associated with a directed polymer in a random medium. Using the Brownian scaling property, we can compute the associated *free energy density*:

$$f(\beta) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta) \tag{40}$$

$$= \gamma(-\beta^2) - 2\log\beta \tag{41}$$

$$= -(-\Psi)^*(-\beta^2) - 2\log\beta. \tag{42}$$

The directed polymer model we have discussed here, and for which we have computed the free energy density, is a continuous version of the classical two-dimensional directed polymer, where it is not known how to compute the free energy density (see, for example, Derrida, 1990).

Comparing the definition of f with (29), by Laplace's method, we expect

$$\lim_{\beta\to\infty} f(\beta)/\beta = 2$$

and this is indeed the case. Note that, equivalently,

$$\lim_{x\to\infty} \gamma(-x)/\sqrt{x} = 2.$$

Thus, we have an alternative strategy for computing the limiting constant in (29).

6. Some remarks on the random matrix connection

Let $\{s(i,j),\ (i,j)\in\mathbb{Z}^2\}$ be a collection of i.i.d. exponential random variables with mean one, and set

$$D(m,n)=\max_{\phi\in\Pi(m,n)}\sum_{(i,j)\in\phi}s(i,j),$$

where $\Pi(m,n)$ is the set of non-decreasing connected paths

$$(1,1)=(i_1,j_1)\leqslant(i_2,j_2)\leqslant\cdots\leqslant(i_{m+n},j_{m+n})=(m,n).$$

This random variable appears in certain growth models and also has an interpretation in terms of $\cdot/M/1$ queues in tandem. It was observed by Johansson (2000) that $D(m,n)$ has the same law as the largest eigenvalue $\rho(m,n)$ of the random matrix AA^*, where A is a $m\times n$ matrix with i.i.d. standard complex normal entries. This random matrix ensemble is known as the *Laguerre*, or *Wishart*, ensemble. Now, by Donsker's theorem, $[D(m,n)-m]/\sqrt{m}$ converges in law, as $m\to\infty$, to the random variable M_n defined by (28). Combining these facts with the observation of Baryshnikov (2001) and Gravner et al. (2000) that M_n has the same law as the largest eigenvalue $\omega(n)$ of an n-dimensional GUE random matrix, we see that the sequence $[\rho(m,n)-m]/\sqrt{m}$ converges in law, as $m\to\infty$, to $\omega(n)$. This complements recent work of Johnstone (2000).

We now give a direct proof of the fact that M_n has the same law as $\omega(n)$, in the case $n=2$. This turns out again to be closely related to Pitman's representation theorem (Pitman, 1975), and in fact there is a process version. Let $B^{(1)}$ and $B^{(2)}$ be independent standard real-valued Brownian motions, and set

$$M_2(t)=\sup_{0<s<t}[B_s^{(1)}+B_{(s,t)}^{(2)}],\tag{43}$$

$$N_2(t)=\inf_{0<s<t}[B_s^{(2)}+B_{(s,t)}^{(1)}].\tag{44}$$

Let X_{11} and X_{22} be independent standard Brownian motions, let $\sqrt{2}X_{12}$ be a standard complex Brownian motion,[1] independent of X_{11} and X_{22}, and set $X_{21}=X_{12}^*$. For each $t>0$, denote the eigenvalues of the 2×2 matrix $X(t)$ by $\lambda_1(t)>\lambda_2(t)$. Finally, let B be a real-valued Brownian motion and R an independent three-dimensional Bessel process.

Theorem 10. *The processes* (M_2,N_2) *and* (λ_1,λ_2) *have the same law, identical to that of* $(B+R,B-R)/\sqrt{2}$.

Proof. Set $\beta^{(1)}=(B^{(1)}-B^{(2)})/\sqrt{2}$ and $\beta^{(2)}=(B^{(1)}+B^{(2)})/\sqrt{2}$. These are independent Brownian motions. Now observe that

$$\sqrt{2}M_2(t)=2\sup_{0<s<t}\beta_s^{(1)}-\beta_t^{(1)}+\beta_t^{(2)}$$

[1] That is, the real and imaginary parts of $\sqrt{2}X_{12}$ are independent standard real-valued Brownian motions.

and

$$\sqrt{2}N_2(t) = -2 \sup_{0<s<t} \beta_s^{(1)} + \beta_t^{(1)} + \beta_t^{(2)},$$

so, by Pitman's theorem, $\sqrt{2}(M_2, N_2)$ has the same law as $(B+R, B-R)$. On the other hand, the eigenvalues of X are given by

$$\sqrt{2}\lambda = \frac{X_{11}+X_{22}}{\sqrt{2}} \pm \sqrt{\left(\frac{X_{11}-X_{22}}{\sqrt{2}}\right)^2 + 2|X_{12}|^2}$$

and the result follows from the independence of $X_{11} - X_{22}$ and $X_{11} + X_{22}$, and the definition of a three-dimensional Bessel process (as the norm of a three-dimensional Brownian motion). □

A multi-dimensional analogue of Theorem 10 is obtained in O'Connell and Yor (2001), which in particular yields a proof of the identity in law observed in Baryshnikov (2001), Gravner et al. (2000).

We remark that McKean (2001) has discussed the law of the eigenvalues of a two-dimensional *Gaussian Orthogonal* matrix:

$$\begin{pmatrix} B_{11} & B_{12}/\sqrt{2} \\ B_{12}/\sqrt{2} & B_{22} \end{pmatrix},$$

where B_{11}, B_{22} and B_{12} are independent standard real-valued Brownian motions. In this case, the eigenvalues are given by

$$\sqrt{2}\lambda = \frac{B_{11}+B_{22}}{\sqrt{2}} \pm \sqrt{\left(\frac{B_{11}-B_{22}}{\sqrt{2}}\right)^2 + B_{12}^2} \tag{45}$$

$$= \beta \pm R^{(2)}, \tag{46}$$

where β is a standard Brownian motion and $R^{(2)}$ is a two-dimensional Bessel process, independent of β. This leads us to recall the following representation of $R_t^{(2)}$, presented in Carmona et al. (1999), namely that for each $t > 0$,

$$\sup_{0<r<s<t} [B_r + B_{(s,t)}]$$

has the same law as $R_t^{(2)}$, where B is a standard Brownian motion. Note that the analogy with Theorem 10 does not extend further: we do not have identity in law as processes, since, from Lévy's representation of reflecting Brownian motion,

$$\left(\sup_{s\leqslant t}|B_s| - |B_t|\right) + L_t,$$

is distributed as the process

$$\sup_{0<r<s<t} [B_r + B_{(s,t)}],$$

which is obviously *transient*, whereas $R^{(2)}$ is recurrent.

Finally, we note that in McKean (2001) it is shown that the joint distribution of the eigenvalues in the Gaussian Orthogonal case cannot be reduced by time and scale change to two-dimensional Brownian motion. In the *Gaussian Unitary* case, which is the case we have discussed here, there is the same impossibility.

7. The role of reversibility

Let $(X_t,\ t \in \mathbb{R})$ be a real-valued stochastic process with $X_0 = 0$ almost surely, and assume that the integral

$$A_t = \int_{-\infty}^{t} \mathrm{d}s \exp 2(X_s - X_t) \tag{47}$$

exists and is finite almost surely for each $t \in \mathbb{R}$. Define a new process $(\hat{X}_t,\ t \in \mathbb{R})$ by

$$\hat{X}_t = X_t + \log(A_t/A_0).$$

Theorem 11. *The process A is stationary and reversible if, and only if, X has stationary and reversible increments and $\hat{X}$ has the same law as X.*

Proof. This theorem is an immediate consequence of the following elementary lemma. □

Lemma 12. *For each $t \in \mathbb{R}$, almost surely,*

$$A_t = \int_{t}^{\infty} \mathrm{d}s \exp 2(\hat{X}_t - \hat{X}_s) \tag{48}$$

and

$$\exp(2X_t) = \frac{A_0}{A_t} \exp\left(\int_0^t \frac{\mathrm{d}s}{A_s}\right). \tag{49}$$

Recall that the formula (48) was presented in Section 3, proof of Theorem 6, in the case where X is Brownian motion with drift; the proof given there applies for general X. The formula (49) can be obtained by elementary calculus. If X has stationary and reversible increments and $\hat{X}$ has the same law as X, then A is stationary by construction and by the formula (48) it is reversible. We now prove the converse. Suppose that A is stationary and reversible. Then, by (49), X has stationary reversible increments. Now by (48),

$$A_{-t} = \int_{-\infty}^{t} \mathrm{d}s \exp 2(\hat{X}_{-t} - \hat{X}_{-s})$$

and hence, by (49),

$$\exp(2\hat{X}_{-t}) = \frac{A_0}{A_{-t}} \exp\left(\int_0^t \frac{\mathrm{d}s}{A_{-s}}\right).$$

The fact that $\hat{X}$ has the same law as X now follows from the fact that A is reversible and X has reversible increments. □

Theorem 11 can be applied to give an alternative proof of Theorem 6, and hence Theorem 3. If X is a standard Brownian motion with positive drift, then $\hat{X}$ has the same law as X: this was proved in Matsumoto and Yor (2001) using the method of enlargement. By Theorem 7, we see that it follows from (and is in fact equivalent to) the reversibility of A, which can be proved directly as in Donati-Martin et al. (2001).

Note that, in the proof of Theorem 11, we use the fact that X can be recovered from A via the formula (49). If, for example, $\log\int\exp$ is replaced by sup, this is no longer the case and the statement of the theorem is false in general. The problem is that the analogue of the formula (49) does not necessarily yield a unique solution to (49). However, all we actually need is that the law of X is uniquely determined by the law of A, which is often the case. (This is certainly the case if X is Brownian motion.) Similar remarks apply if A is defined by

$$A_t = \int_{-\infty}^{t} \mathrm{d}\eta_s \exp 2(X_s - X_t), \tag{50}$$

where η is a random process; in the case where η and X are independent Lévy processes, A is a generalised Ornstein-Uhlenbeck process as discussed in Carmona et al. (2001). In this case, the analogue of (48) holds, that is,

$$A_t = \int_{t}^{\infty} \mathrm{d}\eta_s \exp 2(\hat{X}_t - \hat{X}_s),$$

but given A, Eq. (50) does not necessarily have a unique solution.

See Hambly et al. (2001b) for an application of these ideas in a non-Markovian setting.

8. Multi-dimensional extensions

Theorem 6 has the following multi-dimensional extension (see, for example, Matsumoto and Yor, 2001).

Theorem 13. *Let $m \in \mathbb{R}^d$, $m \neq 0$, $b > 0$. Let B be a standard Brownian motion in $\mathbb{R}^d$, indexed by the entire real line, and set $A_t = \int_{-\infty}^{t} \mathrm{d}s \exp[(m, B_s) + bs]$. Then the following identity in law holds*:

$$\left\{ B_t - 2\frac{m}{|m|^2}\log\left(\frac{A_t}{A_0}\right),\ t \geqslant 0 \right\} \overset{(\mathrm{law})}{=} \left\{ B_t - 2\frac{m}{|m|^2} bt,\ t \geqslant 0 \right\}. \tag{51}$$

The corresponding multi-dimensional analogue of Theorem 3 can be deduced from this by Brownian scaling and Laplace's method.

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