Birational RSK correspondence and Whittaker functions

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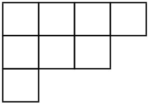
Partitions and tableaux

Let n be a positive integer. A partition $\lambda \vdash n$ is a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ such that $n = \lambda_1 + \lambda_2 + \cdots$.

The *diagram* of a partition λ is a left-justified array of boxes, λ_1 in the first row, λ_2 in the second row, and so on.

For example, the diagram of the partition $(4, 3, 1) \vdash 8$ is



Partitions and tableaux

A *standard tableau* of shape $\lambda \vdash n$ is a filling of (the diagram of) λ with the numbers $1, 2, \ldots, n$ which is increasing across rows and down columns.

A standard tableau with shape $(4, 3, 1) \vdash 8$:

1	3	5	6
2	4	8	
7			

The Robinson-Schensted correspondence

From the representation theory of S_n,

$$n! = \sum_{\lambda \vdash n} d_\lambda^2$$

where $d_\lambda =$ number of standard tableaux with shape λ.

In other words, S_n has the same cardinality as the set of pairs of standard tableaux of size n with the same shape.

Robinson (1938), Schensted (1961): Bijection between S_n and such pairs

$$\sigma \longleftrightarrow (P, Q).$$

The Robinson-Schensted correspondence

For example,

$$356214 \quad \longleftrightarrow \quad \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

Schensted (61): The length of the longest row of P (and Q) equals the length of the longest increasing subsequence in the permutation σ.

The RSK correspondence

Knuth (70): Extends to a bijection between matrices with nonnegative integer entries and pairs of *semi-standard* tableaux of same shape.

A *semistandard tableau* of shape $\lambda \vdash n$ is a filling of λ with positive integers which is *weakly* increasing across rows and strictly increasing down columns.

A semistandard tableau of shape $(5, 3, 1)$:

1	2	2	5	7
3	3	8		
4				

The RSK correspondence

For example,

$$\begin{matrix} 0 & 2 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix} \qquad \longleftrightarrow \qquad \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$$

If $(a_{ij}) \mapsto (P, Q)$, then

$$C_j = \sum_i a_{ij} = \sharp\, j\text{'s in } P$$

$$R_i = \sum_j a_{ij} = \sharp\, i\text{'s in } Q$$

Schur polynomials

For indeterminates $x = (x_1, x_2, \dots, x_r)$ define

$$s_\lambda(x) = \sum_{\text{sh } P = \lambda} x^P,$$

where the sum is over semistandard tableaux P of shape λ, and

$$x^P = x_1^{\sharp 1's\ in\ P} x_2^{\sharp 2's\ in\ P} \dots x_r^{\sharp r's\ in\ P}.$$

E.g.,

$$s_{2,1}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2.$$

Schur polynomials

The Schur polynomials $s_\lambda(x)$, $\lambda \vdash n$, form a $\mathbb{Z}$-basis for homogeneous symmetric polynomials in $x_1, \ldots, x_r$ of degree n, with integer coefficients.

Irreducible characters of $SU(r)$ are given by

$$\chi_\lambda(M) = s_\lambda(e^{i\theta_1}, \ldots, e^{i\theta_r}),$$

for partitions λ with at most $r-1$ parts.

An important application of the Robinson-Schensted correspondence was to prove the Littlewood-Richardson rule, which provides a combinatorial interpretation of the coefficients $c^\nu_{\lambda\mu}$ in the expansion

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu.$$

Cauchy-Littlewood identity

Let $\mathbb{N}^{m\times n} \ni (a_{ij}) \mapsto (P, Q)$ under the RSK correspondence.

Then $C_j = \sum_i a_{ij} = \sharp\, j$'s in P and $R_i = \sum_j a_{ij} = \sharp\, i$'s in Q.

For $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$ we have

$$\prod_{ij} (y_i x_j)^{a_{ij}} = \prod_j x_j^{C_j} \prod_i y_i^{R_i} = x^P y^Q.$$

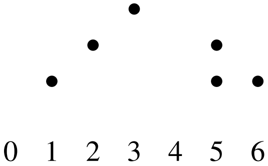
Summing over (a_{ij}) on the left and (P, Q) on the right gives

$$\prod_{ij} (1 - x_i y_j)^{-1} = \sum_\lambda s_\lambda(x) s_\lambda(y).$$

Tableaux and Gelfand-Tsetlin patterns

Semistandard tableaux $\longleftrightarrow$ discrete Gelfand-Tsetlin patterns

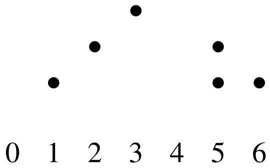
1	1	1	2	2	3
2	2	3	3	3	
3					

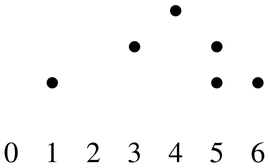


Pairs of tableaux and reverse plane partitions

1	1	1	2	2	3
2	2	3	3	3	
3					

1	1	1	1	2	3
2	2	2	3	3	
3					





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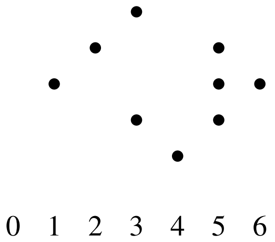

Pairs of tableaux and reverse plane partitions

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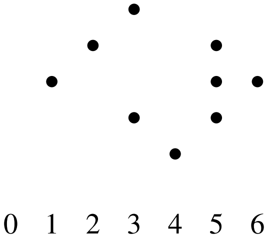


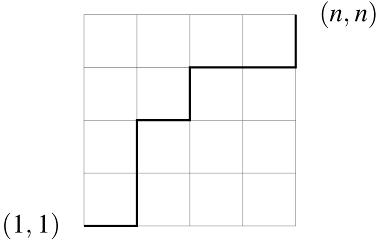
1	2	3
3	5	5
4	5	6

RSK as a tropical map

Assume $m = n$ for simplicity. Then RSK is a map $B : \mathbb{N}^{n\times n} \to \mathbb{N}^{n\times n}$, $A = (a_{ij}) \mapsto B = (b_{ij})$ where B is a reverse plane partition.

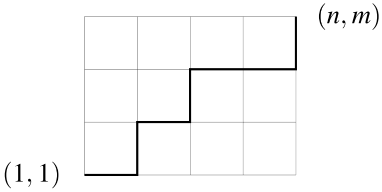
We have the following formulae:

$$b_{nn} = \max_{\phi\in\Pi_{(n,n)}} \sum_{(i,j)\in\phi} a_{ij}$$



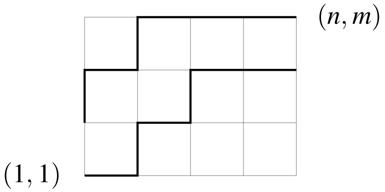
RSK as a tropical map

$$b_{nm} = \max_{\phi \in \Pi_{(n,m)}} \sum_{(i,j) \in \phi} a_{ij}$$



RSK as a tropical map

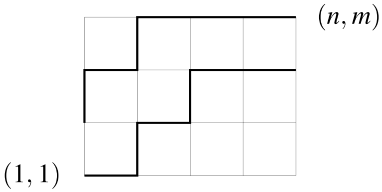
$$b_{n-k+1,m-k+1} + \ldots + b_{nm} = \max_{\phi \in \Pi^{(k)}_{(n,m)}} \sum_{(i,j)\in\phi} a_{ij}$$



RSK as a tropical map

$$b_{n-k+1,m-k+1} + \ldots + b_{nm} = \max_{\phi \in \Pi^{(k)}_{(n,m)}} \sum_{(i,j)\in\phi} a_{ij}$$

$$B(A)' = B(A')$$



Birational RSK correspondence

Replacing these expressions by their $(+, \times)$ counterparts, A.N. Kirillov (00) introduced a *geometric lifting* of RSK correspondence. It is a birational map

$$T : (\mathbb{R}_{>0})^{n\times n} \to (\mathbb{R}_{>0})^{n\times n}$$

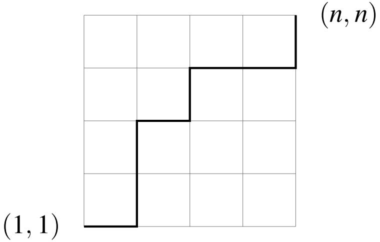
$$X = (x_{ij}) \mapsto (t_{ij}) = T = T(X).$$

For $n = 2$,

$$\begin{matrix} & x_{21} & \\ x_{11} & & x_{22} \\ & x_{12} & \end{matrix} \quad \mapsto \quad \begin{matrix} & x_{11}x_{21} & \\ x_{12}x_{21}/(x_{12}+x_{21}) & & x_{11}x_{22}(x_{12}+x_{21}) \\ & x_{11}x_{12} & \end{matrix}$$

Birational RSK correspondence

$$t_{nn} = \sum_{\phi \in \Pi_{(n,n)}} \prod_{(i,j) \in \phi} x_{ij}$$



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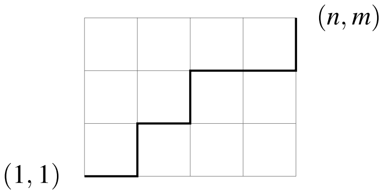
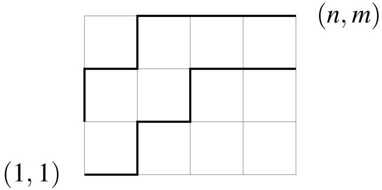


Birational RSK correspondence

$$t_{n-k+1,m-k+1} \dots t_{nm} = \sum_{\phi \in \Pi^{(k)}_{(n,m)}} \prod_{(i,j)\in\phi} x_{ij}$$



Birational RSK correspondence

$$t_{n-k+1,m-k+1} \dots t_{nm} = \sum_{\phi \in \Pi^{(k)}_{(n,m)}} \prod_{(i,j)\in\phi} x_{ij}$$

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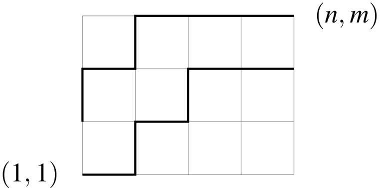

Whittaker functions

- Whittaker functions were first introduced by Jacquet (67). They play an important role in the theory of automorphic forms and also arise as eigenfunctions of the open quantum Toda chain (Kostant 77)

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- The following 'Gauss-Givental' representation for Ψ_λ is due to Givental (97), Joe-Kim (03), Gerasimov-Kharchev-Lebedev-Oblezin (06)

Whittaker functions

A *triangle* P with shape $x \in (\mathbb{R}_{>0})^n$ is an array of positive real numbers:

$$P = \begin{matrix} & & & z_{11} & & & \\ & & z_{22} & & z_{21} & & \\ & \iddots & & & & \ddots & \\ z_{nn} & & & \cdots\cdots & & & z_{n1} \end{matrix}$$

with bottom row $z_{n\cdot} = x$.

Denote by $\Delta(x)$ the set of triangles with shape x.

Whittaker functions

Let

$$P = \begin{array}{ccccccc} & & & z_{11} & & & \\ & & z_{22} & & z_{21} & & \\ & \iddots & & & & \ddots & \\ z_{nn} & & & \cdots & & & z_{n1} \end{array}$$

Define

$$P^\lambda = R_1^{\lambda_1} \left(\frac{R_2}{R_1} \right)^{\lambda_2} \cdots \left(\frac{R_n}{R_{n-1}} \right)^{\lambda_n}, \qquad \lambda \in \mathbb{C}^n, \qquad R_k = \prod_{i=1}^{k} z_{ki}$$

Whittaker functions

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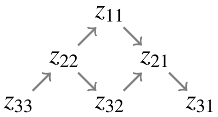
$$\mathcal{F}(P) = \sum_{a \to b} \frac{z_a}{z_b}$$



Whittaker functions

For $\lambda \in \mathbb{C}^n$ and $x \in (\mathbb{R}_{>0})^n$, define

$$\Psi_\lambda(x) = \int_{\Delta(x)} P^{-\lambda} e^{-\mathcal{F}(P)} dP,$$

where $dP = \prod_{1\leq i\leq k<n} dz_{ki}/z_{ki}$.

For $n = 2$,

$$\Psi_{(\nu/2,-\nu/2)}(x) = 2K_\nu\left(2\sqrt{x_2/x_1}\right).$$

These are called $GL(n)$-Whittaker functions.

They are the analogue of the Schur polynomials in the birational setting.

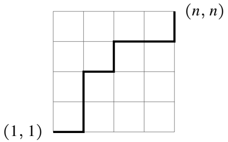
Birational RSK correspondence

Recall

$$X = (x_{ij}) \mapsto (t_{ij}) = T \quad = \quad \begin{matrix} & & t_{31} & & \\ & t_{21} & & t_{32} & \\ t_{11} & & t_{22} & & t_{33} \\ & t_{12} & & t_{23} & \\ & & t_{13} & & \end{matrix}$$

= pair of triangles of same shape $(t_{nn}, \ldots, t_{11})$.

$$t_{nn} = \sum_{\phi \in \Pi_{(n,n)}} \prod_{(i,j)\in\phi} x_{ij}$$



Birational RSK and Whittaker functions

Recall: $X = (x_{ij}) \mapsto (t_{ij}) = T(X) = (P, Q)$.

Theorem (O'C-Seppäläinen-Zygouras 14)

- *The map T is volume-preserving*
- *For $\nu, \lambda \in \mathbb{C}^n$,*

$$\prod_{ij} x_{ij}^{\nu_i+\lambda_j} = P^{\lambda} Q^{\nu}$$

- *The following identity holds:*

$$\sum_{ij} \frac{1}{x_{ij}} = \frac{1}{t_{11}} + \mathcal{F}(P) + \mathcal{F}(Q)$$

Remark - this result is a (significant) refinement of earlier works [O'C '12] and [Corwin-O'C-Seppäläinen-Zygouras '14] where the connection between the geometric RSK correspondence and Whittaker functions was first established.

Analogue of the Cauchy-Littlewood identity

It follows that

$$\prod_{ij} x_{ij}^{-\nu_i-\lambda_j} e^{-1/x_{ij}} \frac{dx_{ij}}{x_{ij}} = P^{-\lambda} Q^{-\nu} e^{-1/t_{11}-\mathcal{F}(P)-\mathcal{F}(Q)} \prod_{ij} \frac{dt_{ij}}{t_{ij}}.$$

Integrating both sides gives, for $\Re(\nu_i + \lambda_j) > 0$:

Corollary (Stade 02)

$$\prod_{ij} \Gamma(\nu_i + \lambda_j) = \int_{\mathbb{R}_+^n} e^{-1/x_n} \Psi_\nu(x) \Psi_\lambda(x) \prod_{i=1}^n \frac{dx_i}{x_i}.$$

This is equivalent to a Whittaker integral identity which was conjectured by Bump (89) and proved by Stade (02). The integral is associated with Archimedean L-factors of automorphic L-functions on $GL(n,\mathbb{R}) \times GL(n,\mathbb{R})$.

Local description

Key ingredient of the proof is a decomposition of the birational RSK map T as a composition of *local* birational maps of the form:

$$\begin{array}{ccc} & b & \\ a & e & d \\ & c & \end{array} \qquad \longrightarrow \qquad \begin{array}{ccc} & b & \\ a & e' & d \\ & c & \end{array}$$

where

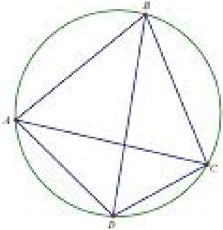
$$ee' = (a+b)\left(\frac{1}{c}+\frac{1}{d}\right)^{-1}.$$

This is related (via a change of variables) to *Ptolemy's relation*.

Ptolemy's relation

Ptolemy's Theorem

The product of the diagonals equals the sum of the products of the two pairs of opposite sides.



$$AC \times BD = (AB \times CD) + (AD \times BC)$$

Symmetric input matrix

Symmetry properties:

$$T(X') = T(X)'$$

$$X \mapsto (P, Q) \qquad \Longleftrightarrow \qquad X' \mapsto (Q, P).$$

$$X = X' \qquad \Longleftrightarrow \qquad P = Q$$

Theorem (O'C-Seppäläinen-Zygouras 14)

The restriction of T to symmetric matrices is volume-preserving.

Symmetric input matrix

The analogue of the Cauchy-Littlewood identity in this setting is:

Corollary

Suppose $s > 0$ and $\Re\lambda_i > 0$ for each i. Then

$$\int_{(\mathbb{R}_{>0})^n} e^{-sx_1}\Psi^n_{-\lambda}(x)\prod_{i=1}^n \frac{dx_i}{x_i} = s^{-\sum_{i=1}^n \lambda_i}\prod_i \Gamma(\lambda_i)\prod_{i<j}\Gamma(\lambda_i+\lambda_j).$$

This is equivalent to a Whittaker integral identity which was conjectured by Bump-Friedberg (90) and proved by Stade (01). The integral is associated with Archimedean L-factors of automorphic L-functions on $GL(n,\mathbb{R})$.

Some things I didn't talk about

- Connections to Toda (integrable systems)
- Connections to random matrices
- Applications to random polymers, KPZ equation, etc.
- q- (and t-) analogues